TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Prof. Dr. Mathias Schulze
Dr. Delphine Pol
Fachbereich Mathematik

Plane Algebraic Curves

Summer Term 2019 - Problem Set 6

Due Date: Friday, July 5, 2019, 10:00 am

Exercise 1. By considering lines passing through convenient points, prove the following results:

(a) Let P be a point on an affine curve F. Show that there is a rational function $\phi \in K(F)$ which has exactly one pole which is simple and at P, i. e. such that $\mu_p(\phi) = -1$ and $\mu_Q(\phi) \geqslant 0$ for all $Q \neq P$.

(b) Let P_1 and P_2 be distinct points on a projective conic F. Show that there is a rational function $\phi \in K(F)$ with $\mu_{P_1}(\phi) = 1$, $\mu_{P_2}(\phi) = -1$, and $\mu_P(\phi) = 0$ at all other points P of F.

Exercise 2. Let P be a point on an affine curve F. Show that there are ring isomorphisms:

(a) $\mathscr{O}_{F,P} \simeq \mathscr{O}_{\mathbb{A}^2,P}/\langle F \rangle$.

(b) $K(F) \simeq K(F^h)$ with

$$K(F) = \left\{ \frac{f}{g} \mid f, g \in A(F) \text{ and } g \neq 0 \right\}$$

and

$$K(F^h) = \left\{ \frac{f}{g} \mid f, g \in S_d(F^h) \text{ for some } d \in \mathbb{N} \text{ and } g \neq 0 \right\}.$$

Exercise 3.

(a) Let F be a projective curve, and let f be a homogeneous polynomial with $\operatorname{div} f = D + E$ for two divisors D and E on F. Show: if D' is linearly equivalent to D (i.e. $D - D' = \operatorname{div}\left(\frac{f_1}{f_2}\right)$ for some homogeneous polynomials of the same degree) and $D' + E$ is effective then there exists $B \in S(F)$ such that $\operatorname{div}(f f_2) = \operatorname{div}(f_1) + \operatorname{div}(B)$. Deduce that $\operatorname{div} B = D' + E$.

(b) Let P, Q, R, S be four distinct points on a cubic curve F. Show that if the intersection point of the lines $\overline{PQ}$ and $\overline{RS}$ lies on F, then $P + Q \sim R + S$.

(c) Show the converse statement: if $P + Q \sim R + S$ then the intersection point of the lines $\overline{PQ}$ and $\overline{RS}$ lies on F. *Hint: use the first question with* $D + E = \operatorname{div}(\overline{PQ})$.

Exercise 4. Let $F = y^2 z - x^3 + xz^2$.

(a) Compute the divisor $\operatorname{div}\left(\frac{y}{z}\right)$ on F.

(b) If ϕ is any non-zero rational function on F whose divisor is the one computed in (a), show that $\phi = \lambda \frac{y}{z}$ for some $\lambda \in K^*$.