TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

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Plane Algebraic Curves

Summer Term 2019 - Problem Set 4

Due Date: Friday, June 7, 2019, 10:00 am

One can use Bézout's theorem for some of the following exercises: If F and G are projective curves without common components over an infinite field K, then $\sum_{P\in F\cap G}\mu_P(F,G)\leqslant \deg(F)\cdot\deg(G)$, with equality if K is algebraically closed.

Exercise 1. Let $d\geqslant 1$ be an integer. We denote by $S_d = K[x,y,z]_d$ the vector space spanned by the monomials of degree d. We recall that $\dim_K(S_d) = \frac{(d+1)(d+2)}{2}$.

(a) Let $r\in\mathbb{N}$ and $P\in\mathbb{P}^2$. We set $S_d(rP) = \{F\in S_d \mid m_P(F)\geqslant r\}$. Compute $\dim_K(S_d(rP))$.

(b) Let $r_1,\ldots,r_n\in\mathbb{N}$ and $P_1,\ldots,P_n\in\mathbb{P}^2$. We set

$$S_d(r_1P_1,\ldots,r_nP_n) = \{F\in S_d \mid \forall i\in\{1,\ldots,n\}, m_{P_i}(F)\geqslant r_i\}.$$

Show that $\dim_K S_d(r_1P_1,\ldots,r_nP_n)\geqslant \frac{(d+1)(d+2)}{2} - \sum_{i=1}^n \frac{r_i(r_i+1)}{2}$.

(c) With the same notations, show that if $\sum_{i=1}^n \frac{r_i(r_i+1)}{2}\leqslant \frac{d(d+3)}{2}$ then there exists a curve (maybe with multiple factors) passing through each of the points P_i with multiplicity at least r_i.

Exercise 2.

(a) Deduce the following real version of Bézout's Theorem from the complex case: If F and G are two real projective curves without common components, then

$$\sum_{P\in F\cap G}\mu_P(F,G) = \deg F\cdot\deg G \mod 2$$

(One can use without proving it that for $P\in\mathbb{P}^2_{\mathbb{R}}$, the intersection multiplicity at P of the real projective curves F and G is the same as the intersection multiplicity at P of F and G as complex curves.)

(b) Deduce that two real projective curves of odd degree have at least one intersection point.

Exercise 3. Let F be a complex irreducible projective curve of degree $d\geqslant 2$, and let $P_0\in\mathbb{P}^2$ be a point. We set $m := m_{P_0}(F)\in\mathbb{N}$.

(a) Let L be a line in $\mathbb{P}^2$ containing P_0. Show that $\mu_{P_0}(F,L)\geqslant m$. When do we have $\mu_{P_0}(F,L) > m$?

(b) Show that there are only finitely many smooth points $P\neq P_0$ of F such that the line passing through P_0 and P is tangent to F at P.

(c) Show that for all but finitely many lines L in $\mathbb{P}^2$ through P_0, the intersection $F\cap L$ consists of exactly $d-m$ points not equal to P_0.

Hint: One can use results from the previous problem sets.

Exercise 4.

(a) Show that a (not necessarily irreducible) reduced projective curve of degree $d\geqslant 2$ over an infinite field has at most $\frac{d(d-1)}{2}$ singular points.

(b) Find an example for each d in which this maximal number of singular points is actually reached.