TECHNISCHE UNIVERSITÄT
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Plane Algebraic Curves

Summer Term 2019 - Problem Set 1

Due Date: Friday, April 26, 2019, 10:00 am

Exercise 1. Consider the curves $F_1 = y - x^2$ and $F_2 = x^3 - y^2$ over K.

(a) Show that the ring $R_1 := K[x, y]/\langle F_1 \rangle$ is isomorphic to $K[t]$.

(b) By considering the intersection of G with a line passing through the origin with arbitrary slope t, show that:

$$V(F_2) = \left\{(t^2, t^3) \in K^2 \mid t \in K\right\}$$

(c) Show that the ring $R_2 := K[x, y]/\langle F_2 \rangle$ is isomorphic to $K[t^2, t^3]$.

(d) Show that the rings R_1 and R_2 are not isomorphic.

Exercise 2.

(a) Draw a picture of the following real curves:

$$F_1 = y^3x^6 - y^6x^2, \quad F_2 = x^2 + y^2 + 2y$$

(b) What are the irreducible components of F_1 and F_2?

(c) Determine the intersection points of F_1 and F_2.

Exercise 3 (Pythagorean triples)**.** Let $F = x^2 + y^2 - 1 \in K[x, y]$ be the unit circle over K . We assume that the characteristic of K is not 2.

(a) Let L be a line passing through $(-1, 0)$ and with slope t. Considering the intersection points of L with F, show that the set of points of F is

$$V(F) = \{(-1, 0)\} \cup \left\{\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \mid t \in K \text{ with } 1+t^2 \neq 0\right\}$$

(b) Prove that the integer solutions (a, b, c) of the equation $a^2 + b^2 = c^2$ are, up to permuting a and b, exactly the triples of the form $\lambda \cdot (u^2 - v^2, 2uv, u^2 + v^2)$ with $\lambda, u, v \in \mathbb{Z}$. *These triples of integers are called Pythagorean triples.*

Exercise 4 (Homogeneous polynomials)**.**

(a) Let $F \in K[x, y]$ be a homogeneous polynomial. We assume that $y-1$ is not an irreducible component of F. Show that the irreducible components and corresponding multiplicities of F are in bijection with those of $F(x, 1) \in K[x]$.

(b) Let $F \in K[x_1, \ldots, x_n]$. Show that F is homogeneous of degree d if and only if for all $\lambda \in K$, $F(\lambda \cdot x_1, \ldots, \lambda \cdot x_n) = \lambda^d \cdot F(x_1, \ldots, x_n)$.