TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Prof. Dr. Mathias Schulze
Raul Epure
Fachbereich Mathematik

Computer Algebra

Summer Term 2019 - Sheet 6

Due Date: Wednesday, **May 29**, 2019, 10:00 am

Exercise 19 (Computing ideal quotients)**.**
Let K be a field of characteristic 0 and let $>$ be a monomial ordering on $\operatorname{Mon}(x_1, \ldots, x_n)$. Consider the two ideals $\mathfrak{a}, \mathfrak{b} \trianglelefteq K[x_1, \ldots, x_n]_>$ and assume that $\mathfrak{a} = \langle g_1, \ldots, g_r \rangle$, $\mathfrak{b} = \langle h_1, \ldots, h_s \rangle$, where $g_i, h_j \in K[x_1, \ldots, x_n]$. Define $h := h_1 + t \cdot h_2 + \ldots + t^{s-1} h_s \in K[t, x_1, \ldots, x_n]$. Prove that

$$\mathfrak{a} : \mathfrak{b} = \langle (\langle g_1, \ldots, g_r \rangle_{K[t,x_1,\ldots,x_n]} : h) \cap K[x_1, \ldots, x_n] \rangle_{K[x_1,\ldots,x_n]_>}.$$

Exercise 20 (Zariski closure of the image)**.**
Consider $\varphi : \mathbb{Q}^2 \to \mathbb{Q}^4$, $(s,t) \mapsto (s^4, s^3t, st^3, t^4)$. Compute the Zariski closure of the image, $\overline{\varphi(\mathbb{Q}^2)}$, and decide whether $\varphi(\mathbb{Q}^2)$ coincides with its closure.

Let R be a commutative ring with unity. An ideal $Q \trianglelefteq R$ is said to be **primary**, if $Q \neq R$ and if for all $x, y \in R$ we have that $xy \in Q$ implies $x \in Q$ or $y^n \in Q$ for some $n \in \mathbb{N}_{\geq 1}$.

Exercise 21. Let R be a commutative ring with unity, $S \subseteq R$ a multiplicative set and $Q \trianglelefteq R$ a primary ideal. Denote by $j : R \to S^{-1}R$ the canonical morphism. Show:

(a) If $S \cap Q \neq \emptyset$, then $S^{-1}Q = S^{-1}R$.

(b) If $S \cap Q = \emptyset$, then $S^{-1}Q$ is a primary ideal of $S^{-1}R$ and $j^{-1}\left(S^{-1}Q\right) = Q$.

(c) If $I \trianglelefteq R$ can be written as $I = \cap_{i=1}^n Q_i$, where the Q_i are primary ideals of R with $S \cap Q_j = \emptyset$ for $j = 1, \ldots, m \leq n$ and $S \cap Q_j \neq \emptyset$ for $j = m+1, \ldots, n$, then the following hold:

   (i) $S^{-1}I = \cap_{i=1}^m S^{-1}Q_i$, and

   (ii) $j^{-1}\left(S^{-1}I\right) = \cap_{i=1}^m Q_i$.

Hint: You may use that localization commutes with finite intersections.