TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

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Computer Algebra

Summer Term 2019 - Sheet 11

Due Date: Thursday, July 4, 2019, 10:00 am

A goal of this exercise sheet is to compute and work with the so called Ext-functor. It is defined as follows:
Let R be a commutative ring with unity and let M, N be R-modules. Assume N has a free resolution

$$\dots \longrightarrow F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} \dots \xrightarrow{\varphi_1} F_0 \longrightarrow N \longrightarrow 0$$

and M has a free presentation

$$G_1 \xrightarrow{\psi} G_0 \xrightarrow{\pi} M \longrightarrow 0\ .$$

We apply $\operatorname{Hom}(-, M)$ to the free presentation of N and obtain the complex

$$0 \longrightarrow \operatorname{Hom}(N, M) \xrightarrow{\operatorname{Hom}(\varphi_0, 1_{\mathrm{M}})} \dots \xrightarrow{\operatorname{Hom}(\varphi_i, 1_{\mathrm{M}})} \operatorname{Hom}(F_i, M) \xrightarrow{\operatorname{Hom}(\varphi_{i+1}, 1_{\mathrm{M}})} \dots$$

Then we can define

$$\operatorname{Ext}^0_R(N, M) := \operatorname{Hom}(N, M) \text{ and } \operatorname{Ext}^i_R(N, M) := \operatorname{Ker}\left(\operatorname{Hom}(\varphi_{i+1}, 1_{\mathrm{M}})\right) / \operatorname{Im}\left(\operatorname{Hom}(\varphi_i, 1_{\mathrm{M}})\right).$$

One can show, that the Ext-functor is independent of the free resolution of N.

Exercise 36. We keep the setup as above. Consider the following commutative diagram with exact columns and exact second and third rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \operatorname{Hom}(F_{i-1}, M) & \xrightarrow{\operatorname{Hom}(\varphi_i, 1_{\mathrm{M}})} & \operatorname{Hom}(F_i, M) & \xrightarrow{\operatorname{Hom}(\varphi_{i+1}, 1_{\mathrm{M}})} & \operatorname{Hom}(F_{i+1}, M) & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \operatorname{Hom}(F_{i-1}, G_0) & \xrightarrow{\operatorname{Hom}(\varphi_i, \mathrm{id}_{G_0})} & \operatorname{Hom}(F_i, G_0) & \xrightarrow{\operatorname{Hom}(\varphi_{i+1}, \mathrm{id}_{G_0})} & \operatorname{Hom}(F_{i+1}, G_0) & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow{\scriptstyle \operatorname{Hom}(\mathrm{id}_{F_i}, \psi)} & & \uparrow{\scriptstyle \operatorname{Hom}(\mathrm{id}_{F_{i+1}}, \psi)} & & \\
\dots & \longrightarrow & \operatorname{Hom}(F_{i-1}, G_1) & \xrightarrow{\operatorname{Hom}(\varphi_i, \mathrm{id}_{G_1})} & \operatorname{Hom}(F_i, G_1) & \xrightarrow{\operatorname{Hom}(\varphi_{i+1}, \mathrm{id}_{G_1})} & \operatorname{Hom}(F_{i+1}, G_1) & \longrightarrow & \dots
\end{array}$$

Prove that for $i \geq 1$:

$$\operatorname{Ext}^i_R(N, M) \cong \operatorname{Hom}(\varphi_{i+1}, \mathrm{id}_{G_0})^{-1}\left(\operatorname{Im}\left(\operatorname{Hom}(\mathrm{id}_{F_{i+1}}, \psi)\right)\right) / \left(\operatorname{Im}\left(\operatorname{Hom}(\mathrm{id}_{F_i}, \psi)\right) + \operatorname{Im}\left(\operatorname{Hom}(\varphi_i, \mathrm{id}_{G_0})\right)\right).$$

Hint: Have a look at the proof of Proposition 7.1.3 in the book by Greuel and Pfister.

Exercise 37. We keep the setup and notation as before. State an algorithm similar to Example 7.1.5 in the book by Greuel and Pfister to compute $\operatorname{Ext}^i_R(N, M)$ for $i \geq 1$ and prove its correctness.
Hint: For commands regarding the computation of Hom maps, see Example 2.1.26 in the book by Greuel and Pfister.

Exercise 38. Prove the following:

(a) There exist modules N and M, such that $\operatorname{Ext}^i_R(N, M) \neq \operatorname{Ext}^i_R(M, N)$.

(b) If N is a free module, then $\operatorname{Ext}^i_R(N, M) = 0$ for all $i \geq 1$.

(c) If R is a principal ideal domain, then $\operatorname{Ext}^i_R(N, M) = 0$ for all $i \geq 2$.

Exercise 39. Let R be a Noetherian commutative ring with unity and $M \neq \{0\}$ a finitely generated R-module. Recall that $P \in \operatorname{Ass}(M)$ if and only if there exists an $m \in M$, such that $P = 0 : \langle m \rangle = \operatorname{Ann}(m)$. Prove the following:

(a) $M^{\text{reg}} = \{r \in R \mid r \cdot m \neq 0 \text{ for all } m \in M \setminus \{0\}\} = R \setminus \bigcup_{P \in \operatorname{Ass}(M)} P$.

(b) There exists a chain of submodules

$$M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_{n-1} \supseteq M_n = 0,$$

and prime ideals $P_0, \ldots, P_{n-1} \in \operatorname{Spec}(R)$, such that

$$M_i / M_{i+1} = R / P_i$$

for $i = 0, \ldots, n-1$.

Hint for (b): Choose a $P \in \operatorname{Ass}(M)$ and let $P = 0 : \langle m \rangle$. If $M = \langle m \rangle$ then $M \cong R/P$, otherwise continue with $M / \langle m \rangle$.