TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

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Computer Algebra

Summer Term 2019 - Sheet 1

Due Date: Thursday, April 25, 2019, 10:00 am

Exercise 1. Let A be a ring and $f = \sum_{|\alpha|\geq 0} a_\alpha x^\alpha \in A[x_1,\ldots,x_n]$. Prove the following statements:

(a) f is nilpotent if and only if a_α is nilpotent for all α. In particular: $A[x_1,\ldots,x_n]$ is reduced if and only if A is reduced.
(Hint: Argue by induction on the number of variables.)

(b) f is a unit in $A[x_1,\ldots,x_n]$ if and only if $a_{(0,\ldots,0)}$ is a unit in A and a_α are nilpotent for $\alpha \neq (0,\ldots,0)$. In particular: $(A[x_1,\ldots,x_n])^* = A^*$ if and only if A is reduced.
(Hint: First prove the following statement using a geometric series: If $a \in A[x_1,\ldots,x_n]$ is a unit and $b \in A[x_1,\ldots,x_n]$ is nilpotent, then $a+b$ is a unit. Deduce the "if"-part from this statement. For the "only if"-part, use this statement and induction on the number of variables.)

Exercise 2. Let A be a ring and $f = \sum_{|\alpha|\geq 0} a_\alpha x^\alpha \in A[x_1,\ldots,x_n]$. Prove the following statements:

(a) f is a zero-divisor in $A[x_1,\ldots,x_n]$ if and only if there exisits some $a \neq 0$ in A such that $af = 0$. In particular: $A[x_1,\ldots,x_n]$ is an integral domain if and only if A is an integral domain.

(b) $A[x_1,\ldots,x_n]$ is an integral domain if and only if $\deg(fg) = \deg(f)+\deg(g)$ for all $f,g \in A[x_1,\ldots,x_n]$.

Exercise 3. Monomial orderings arise in a variety of ways. One possibility is to use matrices to define monomial orderings: The matrix $A \in \mathrm{GL}(n,\mathbb{R})$ defines a monomial ordering $>_A$ on $\mathrm{Mon}(x_1,\ldots,x_n)$ by setting

$$x^\alpha >_A x^\beta :\Leftrightarrow A\alpha >_{\text{lex}} A\beta,$$

where $>_{\text{lex}}$ on the right–hand side is the lexicographical ordering on $\mathbb{R}^n$. One can also define new monomial orderings from "known" orderings using so-called **product orderings**: Consider a monomial ordering $>_1$ on $\mathrm{Mon}(x_1,\ldots,x_{n_1})$ and a monomial ordering $>_2$ on $\mathrm{Mon}(y_1,\ldots,y_{n_2})$. Then the product ordering or block ordering $>$, also denoted by $(>_1,>_2)$, on $\mathrm{Mon}(x_1,\ldots,x_{n_1},y_1,\ldots,y_{n_2})$, is defined as

$$x^\alpha y^\beta > x^{\alpha'} y^{\beta'} :\Leftrightarrow x^\alpha >_1 x^{\alpha'} \text{ or } \left(x^\alpha = x^{\alpha'} \text{ and } y^\beta >_2 y^{\beta'}\right).$$

Given a vector $w = (w_1,\ldots,w_n)$ of integers, we define the weighted degree of x^α by

$$w\text{-}\deg(x^\alpha) := \langle w,\alpha\rangle := w_1\alpha_1 + \cdots + w_n\alpha_n,$$

that is, the variable x_i has degree w_i. For a polynomial $f = \sum_\alpha a_\alpha x^\alpha$, we define the weighted degree,

$$w\text{-}\deg(f) := \max\{w\text{-}\deg(x^\alpha) \mid a_\alpha \neq 0\}.$$

Using the weighted degree in the definition of $>_{dp}$, respectively $>_{ds}$ (cf. Example 1.2.8 in the SINGULAR book by Greuel, Pfister), with all $w_i > 0$, instead of the usual degree, we obtain the **weighted reverse lexicographical ordering** $>_{wp(w_1,\ldots,w_n)}$, respectively the **negative weighted reverse lexicographical ordering** $>_{ws(w_1,\ldots,w_n)}$.

(a) Show that $>_A$ is indeed a monomial ordering on $\mathrm{Mon}(x_1,\ldots,x_n)$.

(b) Determine matrices $A \in \mathrm{GL}(n,\mathbb{R})$ defining the orderings

   (i) $>_{ws(5,3,4)}$ on $\mathrm{Mon}(x_1,x_2,x_3)$ with $n = 3$,

   (ii) $(>_{dp}, >_{ls})$ on $\mathrm{Mon}(x_1,\ldots,x_{n_1},y_1,\ldots,y_{n_2})$ with $n = n_1 + n_2$,

   (iii) $(>_{ds}, >_{wp(7,1,9)})$ on $\mathrm{Mon}(x_1,\ldots,x_{n_1},y_1,y_2,y_3)$ with $n = n_1 + 3$.

Exercise 4. Write a SINGULAR procedure `pairSet(list P,ideal I, poly f)`, having a list $P = ((g_1,h_1),\ldots,(g_r,h_r))$ of pairs of polynomials, an ideal $I = \langle f_1,\ldots,f_s\rangle$ and a polynomial f as input and returning the extended pair set $P = P \cup ((f,f_1),\ldots,(f,f_s))$ as output.

Don't forget to add at least one example to your procedure!