TECHNISCHE UNIVERSITÄT
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Commutative Algebra

Winter Semester 2016 - Problem Set 9

Due January 13, 2017, 1 p.m.

Problem 1: Let R be a ring. Show:

(a) Let M be an R-module. If $P \leq M$ is primary then

$$\sqrt{\operatorname{ann}(M/P)} = \{r \in R \mid r\cdot \in \operatorname{End}_R(M/P) \text{ not injective}\} \in \operatorname{Spec}(R).$$

(b) $Q \lneq R$ is a primary ideal if and only if

$$\forall r, s \in R\colon r \cdot s \in Q \implies s \in Q \vee r \in \sqrt{Q}.$$

(c) Let $I \trianglelefteq R$ with $\sqrt{I} \in \operatorname{Max}\operatorname{Spec}(R)$. Then I is $\sqrt{I}$-primary. In particular $\mathfrak{m}^n$ is $\mathfrak{m}$-primary for $\mathfrak{m} \in \operatorname{Max}\operatorname{Spec}(R)$ and $n \in \mathbb{N}_{\geq 1}$.

Problem 2: Let K be a field. Compute two different minimal primary decompositions of the ideal $I = \langle XY, Y^2 \rangle \trianglelefteq K[X,Y]$.

Problem 3: Let R be a noetherian ring and M a finitely generated R-module. Let $N \leq M$ be a submodule with minimal primary decomposition $N = \bigcap_{i=1}^n P_i$, where P_i is a $\mathfrak{p}_i$-primary submodule of M for $i = 1, \ldots, n$. Show:

(a) $\sqrt{\operatorname{ann}(M/N)} = \bigcap_{i=1}^n \mathfrak{p}_i$.

(b) $\dim(M/N) = \max_{1 \leq i \leq n} \dim(M/P_i)$.

Problem 4: Let $R = \bigoplus_{i \in \mathbb{N}} R_i$ be a graded ring, $M = \bigoplus_{i \in \mathbb{Z}} M_i$ a graded R-module and $\mathfrak{p} \in \operatorname{Ass} M$. Show:

(a) $\mathfrak{p}$ is a graded ideal of R.

(b) $\mathfrak{p}$ is the annihilator of a homogeneous element of M.