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Commutative Algebra

Winter Semester 2016 - Problem Set 8

Due January 6, 2017, 1 p.m.

Problem 1: Let $R = \bigoplus_{i\in\mathbb{N}} R_i$ be a graded ring, $M = \bigoplus_{i\in\mathbb{Z}} M_i$ a graded R-module and $N \leq M$ a submodule. We denote elements of M by $\sum_{i\in\mathbb{Z}} m_i := (m_i)_{i\in\mathbb{Z}} \in M$. Each $m \in M_i$, $i \in \mathbb{Z}$, is called a homogeneous element of M. We say that N is a graded R-submodule of M if $N = \bigoplus_{i\in\mathbb{Z}}(N \cap M_i)$. Show that the following are equivalent:

(a) N is a graded R-submodule.

(b) $N = \sum_{i\in\mathbb{Z}}(N \cap M_i)$.

(c) If $n = (n_i)_{i\in\mathbb{Z}} \in N$, then $n_i \in N$ for all i.

(d) There exists a set of homogeneous elements of M generating N.

Problem 2: Let K be a field and $R = K[X,Y]/\left\langle X^2Y^3, X^3Y^2, X^4Y\right\rangle$.

(a) Determine candidates for associated primes of R using a monomial diagram as explained in the lecture. Show that your candidates are indeed associated primes of R.

(b) Compute a prime filtration of R to show that you have found all associated primes in part (a).

(c) (extra credit) Let $I \trianglelefteq K[X,Y]$ be any monomial ideal (i.e. an ideal generated by monomials). Which sets of associated primes of $K[X,Y]/I$ are possible and under what conditions on I?

Problem 3: Let R be a ring, M a finitely presented R-module and N any R-module. Show that

$$\operatorname{Ass}(\operatorname{Hom}_R(M,N)) = \operatorname{Supp}(M) \cap \operatorname{Ass}(N).$$

Hint: Proceed as follows:

(a) Show that $\operatorname{Ass}(P) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{Hom}_R(R/\mathfrak{p}, P)_\mathfrak{p} \neq 0\}$ *for any* R*-module* P.

(b) Show that $\operatorname{Hom}_R(P, \operatorname{Hom}_R(P', P'')) \cong \operatorname{Hom}_R(P', \operatorname{Hom}_R(P, P''))$ *for any* R*-modules* P, P', P'' *by applying isomorphisms from the lecture.*

(c) Use part (b), Remark 3.33 and Remark 3.10(a) to show that $\operatorname{Hom}_R(R/\mathfrak{p}, \operatorname{Hom}_R(M,N))_\mathfrak{p} \cong \operatorname{Hom}_{\kappa_{R_\mathfrak{p}}}(M_\mathfrak{p} \otimes_{R_p} \kappa_{R_\mathfrak{p}}, \operatorname{Hom}(R/\mathfrak{p}, N)_\mathfrak{p})$.

(d) Apply parts (a) and (c) to prove the claim.

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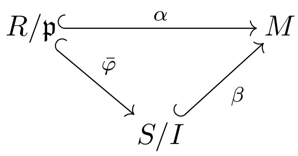
Please wait until December 20 to solve Problem 4 (you need Lemma 5.15 and Corollary 5.16 for its solution).

Problem 4: Let R be a ring, S a noetherian ring, $\varphi \in \operatorname{Hom}(R, S)$ and M an S-module. Show that

$$\varphi^{\#}(\operatorname{Ass}_S(M)) = Ass_R(M)$$

(where $\varphi^{\#}$ was defined in Remark 1.19(e)).
Hint: For the inclusion $\supseteq$ proceed as follows: Construct for $\mathfrak{p} \in Ass_R(M)$ an injective homomorphism $\alpha : R/\mathfrak{p} \hookrightarrow M$. Find $I \trianglelefteq S$ and a homomorphism β such that the diagram



commutes and all involved maps are injective. Show that the composed map

$$\psi : R/\mathfrak{p} \hookrightarrow S/I \twoheadrightarrow (S/I)^{red} \cong S/\sqrt{I}$$

is injective. Apply Corollary 4.34 to get that the composed map

$$\psi' : S/\sqrt{I} \hookrightarrow \prod_{\mathfrak{q}/\sqrt{I} \in \operatorname{Min}\operatorname{Spec}(S/\sqrt{I})} (S/\sqrt{I})/(\mathfrak{q}/\sqrt{I}) \cong \prod_{\mathfrak{q}/I \in \operatorname{Min}\operatorname{Spec}(S/I)} S/\mathfrak{q}$$

is injective. Deduce from the injectivity of the map $\psi' \circ \psi : R/\mathfrak{p} \hookrightarrow \prod_{\mathfrak{q}/I \in \operatorname{Min}\operatorname{Spec}(S/I)} S/\mathfrak{q}$ that there is $\mathfrak{q}/I \in \operatorname{Min}\operatorname{Spec}(S/I)$ such that $\bar{\varphi}^{-1}(\mathfrak{q}/I) = 0$ and hence $\mathfrak{q}^c = \mathfrak{p}$. Show that $\mathfrak{q} \in \operatorname{Min}\operatorname{Supp}_S(S/I)$ and apply Lemma 5.15 and Lemma 5.8(a) to finish your proof.