TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

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Commutative Algebra

Winter Semester 2016 - Problem Set 7

Due December 16, 2016, 1 p.m.

Problem 1: Let M be an R-module. We define the *socle* of M by

$$\operatorname{soc}(M) := \sum\{N \mid N \leq M \text{ simple}\}.$$

We say that a submodule $N \leq M$ is an *essential* submodule of M, and write $N \subseteq_e M$, if for every submodule $N' \leq M$,

$$N' \cap N = 0 \text{ implies } N' = 0.$$

Show:

(a) $\operatorname{soc}(M) = \bigcap\{N \mid N \subseteq_e M\}$.

(b) If M is artinian, then $\operatorname{soc}(M) \subseteq_e M$.

(c) If $(R, \mathfrak{m})$ is local, then $\operatorname{soc}(M) = \{m \in M \mid \mathfrak{m} \cdot m = 0\}$.

Problem 2: Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local ring homomorphism, and N an R-flat S-module such that $N/\mathfrak{m}N$ has finite length over S (i.e. $\ell_S(N/\mathfrak{m}N) < \infty$) . Show that for every finite length R-module M

$$\ell_S(M \otimes_R N) = \ell_R(M) \cdot \ell_S(N/\mathfrak{m}N).$$

Hint: Use induction on $\ell_R(M)$.

Problem 3:

(a) Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{m}$ minimal over $I \trianglelefteq R$. Show that $\sqrt{I} = \mathfrak{m}$.

(b) Let R be a ring and $\mathfrak{p} \in \operatorname{MinSpec}(R)$. Show: If R_p is reduced, then

$$R_\mathfrak{p} \cong Q(R/p)$$

is a field.

(c) Let R be a reduced ring with $|\operatorname{MinSpec}(R)| < \infty$ and $U \subseteq R$ multiplicatively closed. Show that $U^{-1}Q(R) \cong Q(U^{-1}R)$.

Problem 4:
Let $U \subseteq R$ be multiplicatively closed, M an R-module, $N \leq_R M$, $P \leq_{U^{-1}R} U^{-1}M$. We use exactness of localization to identify $U^{-1}N$ with a submodule of $U^{-1}M$. Show:

(a) $U^{-1}N = U^{-1}R \cdot \iota_M(N)$ and $\iota_M^{-1}(P) = \left\{m \in M \mid \frac{m}{1} \in P\right\}$.

(b) $\iota_M^{-1}(U^{-1}N) = \{m \in M \mid \exists u \in U \colon um \in N\}$ and $U^{-1}\iota_M^{-1}(P) = P$.