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Commutative Algebra

Winter Semester 2016 - Problem Set 6

Due December 9, 2016, 1 p.m.

Problem 1: An R-module M is called *semisimple* if it satisfies the following equivalent conditions:

(i) M is a sum of simple modules.

(ii) M is a direct sum of simple modules.

(iii) For every submodule $N \leq M$ there is a module P such that $M = N \oplus P$.

(a) Show that condition (i) implies condition (ii).
Hint: Assume $M = \sum_{i \in I} M_i$. Use Zorn's lemma to find a maximal $J \subseteq I$ such that $\sum_{j \in J} M_j = \bigoplus_{j \in J} M_j$.

(b) Let M and N be R-modules. Show that if M or N is semisimple, then $M \otimes_R N$ is semisimple. Is the converse also true?
Hint: Reduce to the case M simple and show that in this case $M \otimes_R N$ is a semisimple vectorspace.

Problem 2: Let M be an R-module. Show:

(a) The following are equivalent:

  (i) M is noetherian.

  (ii) Every non-empty set of submodules of M has a maximal element (with respect to inclusion).

  (iii) Every non-empty set of finitely generated submodules of M has a maximal element.

  If we replace "noetherian" by "artinian" and "maximal" by "minimal" which of the above statements are still equivalent?

(b) If $N_1, N_2 \leq M$ are submodules such that M/N_1 and M/N_2 are noetherian/artinian, then $M/(N_1 \cap N_2)$ is noetherian/artinian.

Problem 3: Let M be an artinian R-module and $\varphi : M \to M$ an R-module homomorphism. Show that φ is an isomorphism if it is injective.
Hint: Consider the morphisms φ^n for $n \in \mathbb{N}$.

Problem 4: Let M be a noetherian R-module and let $I = \operatorname{ann}_R(M)$. Prove that R/I is a noetherian ring. Is the result still true if we replace "noetherian" by "artinian"?

In-class Problem

(a) Show that $\mathbb{Q}/\mathbb{Z}$ is neither noetherian nor artinian as $\mathbb{Z}$-module.

(b) Let $p \in \mathbb{P}$ and $G := \{q \in \mathbb{Q}/Z \mid \operatorname{ord}(q) = p^n \text{ for some } n \in \mathbb{N}\} \leq \mathbb{Q}/\mathbb{Z}$. Show that G is an artinian $\mathbb{Z}$-module, but not noetherian as a $\mathbb{Z}$-module.