Technische Universität Kaiserslautern

Prof. Dr. Mathias Schulze
Dipl.-Math. Cornelia Rottner
Fachbereich Mathematik

Commutative Algebra

Winter Semester 2016 - Problem Set 4

Due November 25, 2016, 1 p.m.

Problem 1: Let M, N and P be R-modules. Show that:

$$(M \otimes_R N) \otimes_R P \cong M \otimes_R (N \otimes_R P).$$

Problem 2: Show: If M is a finitely presented R-module, N any R-module and P a flat R-module then there is an isomorphism of R-modules

$$\operatorname{Hom}_R(M, N) \otimes_R P \to \operatorname{Hom}_R(M, N \otimes_R P), \quad \varphi \otimes p \mapsto (m \mapsto \varphi(m) \otimes p).$$

Hint: First construct the above map using the universal property of the tensor product. Then prove the claim in the case that M is a free R-module. Finally, apply the functors $\operatorname{Hom}_R(-, N) \otimes_R P$ and $\operatorname{Hom}_R(-, N \otimes_R P)$ to a finite presentation of M.

Problem 3: Let $(R, \mathfrak{m})$ be a local ring, and M and N be finitely generated R-modules. Show that $M \otimes_R N = 0$ if and only if $M = 0$ or $N = 0$.
Hint: Nakayama's lemma.

Problem 4: Let M and N be R-modules, and suppose that $N = \langle n_\lambda \mid \lambda \in \Lambda \rangle$. Show:

(a) $M \otimes_R N = \{\sum_{\lambda \in \Lambda} m_\lambda \otimes n_\lambda \mid m_\lambda \in M \text{ and } m_\lambda \neq 0 \text{ for only finitely many } \lambda \in \Lambda\}$.

(b) Let $x = \sum_{\lambda \in \Lambda} m_\lambda \otimes n_\lambda \in M \otimes_R N$ with $m_\lambda \in M$ and $m_\lambda \neq 0$ for only finitely many $\lambda \in \Lambda$. Then $x = 0$ if and only if, for some index set Θ, there exist $m'_\theta \in M$ and $a_{\lambda,\theta} \in R$ for $\lambda \in \Lambda$ and $\theta \in \Theta$, such that

- $m'_\theta \neq 0$ for only finitely many $\theta \in \Theta$,
- $a_{\lambda,\theta} \neq 0$ for only fintely many $\lambda \in \Lambda$ for each fixed $\theta \in \Theta$,
- $m_\lambda = \sum_{\theta \in \Theta} a_{\lambda,\theta} \cdot m'_\theta$ for all $\lambda \in \Lambda$,
- $\sum_{\lambda \in \Lambda} a_{\lambda,\theta} \cdot n_\lambda = 0$ for all $\theta \in \Theta$.

Hint: First show that if $R^{(\Lambda)} \to N, e_\lambda \mapsto n_\lambda$ is an isomorphism, all m_λ are actually zero. Then consider a free presentation $R^{(\Theta)} \to R^{(\Lambda)} \to N \to 0$ of N and tensorize it with M.