TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

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Commutative Algebra

Winter Semester 2016 - Problem Set 2

Due November 11, 2016, 1 p.m.

All rings are commutative with 1.

Problem 1: Let R be a ring and $r \in R$ idempotent. Show that every prime ideal of R contains either r or $1 - r$.

Problem 2: Let $(R, \mathfrak{m}_R)$ and $(S, \mathfrak{m}_S)$ be local rings and $\varphi : R \to S$ be a ring homomorphism. Show that the following are equivalent:

(a) $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$.

(b) $\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$.

(c) For any $x \in R$, if $\varphi(x)$ is invertible in S, then x is invertible in R.

If φ satisfies the above conditions, we call φ a local ring homomorphism.

Problem 3: Let $\varphi : R \to S$ be a ring homomorphism, M an R-module and N an S-module. Show that

$$\mathrm{Hom}_R(N, M) \cong \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)).$$

Problem 4: Let R be a ring, $I \trianglelefteq R$ nilpotent and N, N', M, M' R-modules with $N, N' \leq M$. Show:

(a) If $IM = M$, then $M = 0$.

(b) If $M = N + IN'$, then $M = N$.

(c) The morphism $\varphi \in \mathrm{Hom}_R(M', M)$ is surjective if $\overline{\varphi} : M'/IM' \to M/IM$ is surjective.

(d) Let Λ be some index set. The set $\{m_\lambda\}_{\lambda \in \Lambda} \subseteq M$ generates M as R-module if $\{\overline{m}_\lambda\}_{\lambda \in \Lambda}$ generates M/IM as R/I-module.