TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Prof. Dr. Mathias Schulze
Dipl.-Math. Cornelia Rottner
Fachbereich Mathematik

Commutative Algebra

Winter Semester 2016 - Problem Set 1

Due November 4, 2016, 1 p.m.

All rings are commutative with 1.

Problem 1: Let R be a ring and $I, J_1, \ldots, J_n \trianglelefteq R$. Show that:

(a) $I : (\sum_{i=1}^n J_i) = \bigcap_{i=1}^n (I : J_i)$.

(b) $(\bigcap_{i=1}^n J_i) : I = \bigcap_{i=1}^n (J_i : I)$.

(c) $\sqrt{J_1 \cap \cdots \cap J_n} = \sqrt{J_1} \cap \cdots \cap \sqrt{J_n}$.

(d) $\sqrt{J_1 + \cdots + J_n} \supseteq \sqrt{J_1} + \cdots + \sqrt{J_n}$.

Problem 2: Let R be a ring, $r \in R$ nilpotent and $u \in R$ a unit. Show that $u + r$ is a unit.

Problem 3: Let R be a ring and $I \trianglelefteq R$. The natural injection $R \hookrightarrow R[\underline{x}] := R[x_1, \ldots, x_n], a \mapsto a$ is a ring homomorphism and thus makes $R[\underline{x}]$ into an R-algebra.

(a) Show that $R[\underline{x}]$ satisfies the following *universal property*: If R' is any R-algebra and $a_1, \ldots, a_n \in R'$ are given, then there is a unique R-algebra homomorphism $\alpha : R[\underline{x}] \to R'$ such that $\alpha(x_i) = a_i$ for $i = 1, \ldots, n$. (I.e., an R-algebra homomorphism on $R[\underline{x}]$ may be uniquely defined by specifying the images of the x_i.)

(b) Let $I \trianglelefteq R[\underline{x}]$ be an ideal. Show that for any R-algebra A there is a canonical bijection

$$\operatorname{Hom}_R(R[\underline{x}]/I, A) \to Z_A(I) := \{(a_1, \ldots, a_n) \in A^n \mid f(a_1, \ldots, a_n) = 0 \, \forall \, f \in I\}.$$

(What is the meaning of $f(a_1, \ldots, a_n)$ here?)

(c) Determine $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[x]/\langle x^2 + 1\rangle, \mathbb{Z})$ and $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[x]/\langle x^2 + 1\rangle, \mathbb{C})$ explicitly.

Problem 4: Let R be a ring. Show that the following are equivalent:

(a) $R/N(R)$ is a field.

(b) $|\operatorname{Spec}(R)| = 1$.

(c) Every element of R is either a unit or nilpotent.

In-class Problem 5: Does the following equality hold in the polynomial ring $\mathbb{C}[x, y]$:

$$\langle x^3 - x^2, x^2y - x^2, xy - y, y^2 - y\rangle = \langle x^2, y\rangle \cap \langle x - 1, y - 1\rangle$$

In-class Problem 6: Determine $\operatorname{Hom}_{\mathbb{C}}(\mathbb{C}[x, y]/\langle xy, x^2 - x\rangle, \mathbb{C}[x, y]/\langle x^2 - y^3\rangle)$.