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Commutative Algebra

Winter Semester 2016 - Problem Set 0

Problem 1: Let R be a principal ideal domain, $m, n \in R \setminus \{0\}$, g a greatest common divisor of m and n, and l a least common multiple of m and n. Show the following properties:

(a) $\langle n \rangle + \langle m \rangle = \langle n, m \rangle = \langle g \rangle$

(b) $\langle n \rangle \cap \langle m \rangle = \langle l \rangle$

(c) $\langle n \rangle \cdot \langle m \rangle = \langle n \cdot m \rangle$

(d) $\langle n \rangle : \langle m \rangle = \left\langle \frac{n}{g} \right\rangle = \left\langle \frac{l}{m} \right\rangle$

(e) $\sqrt{\langle n \rangle} = \langle p_1 \cdots p_k \rangle$ if $p_1, \ldots, p_k \in \mathbb{P}$ are the distinct prime factors of n.

Do the above properties hold also for unique factorization domains?

Problem 2:

(a) Let R be a ring. Show:

(i) $x \in R[\![x]\!]^{\mathrm{reg}}$

(ii) $R[\![x]\!]^* = \{a \in R[\![x]\!] \mid a_0 \in R^*\}$

(iii) $a \in R[\![x]\!]$ nilpotent $\implies \forall i \in \mathbb{N}\colon a_i$ nilpotent. Is the converse also true?

(b) Let K be a field. Show that $K[\![x]\!]$ is a principal ideal domain whose ideals are generated by any element of minimal order.

Problem 3: Let $\varphi : R \to S$ be a homomorphism of rings, $I_1, I_2 \trianglelefteq R$ and $J_1, J_2 \trianglelefteq S$. Show:

(a) $(I_1 + I_2)^e = I_1^e + I_2^e$, $(J_1 + J_2)^c \supseteq J_1^c + J_2^c$;

(b) $(I_1 \cap I_2)^e \subseteq I_1^e \cap I_2^e$, $(J_1 \cap J_2)^c = J_1^c \cap J_2^c$;

(c) $(I_1 I_2)^e = I_1^e I_2^e$, $(J_1 J_2)^c \supseteq J_1^c J_2^c$;

(d) $(I_1 : I_2)^e \subseteq (I_1^e : I_2^e)$, $(J_1 : J_2)^c \subseteq (J_1^c : J_2^c)$;

(e) $(\sqrt{I_1})^e \subseteq \sqrt{I_1^e}$, $(\sqrt{J_1})^c = \sqrt{J_1^c}$.

Show that the above inclusion are in general proper.

Problem 4: Consider the ring homomorphism

$$\mathbb{Z} \to R := \left\{ \frac{z}{7^n} \mid n \geq 0, z \in \mathbb{Z} \right\} \subset \mathbb{Q} : z \mapsto z$$

and the ideals $I = \langle 84 \rangle \trianglelefteq \mathbb{Z}$ and $J = \langle 15 \rangle \trianglelefteq R$. Give generators of I^e, I^{ec}, J^c and J^{ce}.