Technische Universität Kaiserslautern

Prof. Dr. Mathias Schulze
Dipl.-Math. Cornelia Rottner
Fachbereich Mathematik

Computer Algebra

Winter Semester 2014 - Problem Set 7

Due December 17, 2015, 10:00

Problem 1: Let $A = \mathbb{Q}[x,y]_{\langle x,y\rangle}/\langle xy\rangle$. Compute the Betti numbers of the A-module $M = \langle (x^2,y),(x,y)\rangle \leq A^2$ by hand. Use Singular to check your results.

Problem 2: Let R be a local Noetherian ring, let M be a finitely generated R-module, and let $\{f_1,\ldots,f_k\}$, $\{g_1,\ldots,g_k\}$ be two minimal sets of generators. Prove that $\mathrm{syz}(f_1,\ldots,f_k) \cong \mathrm{syz}(g_1,\ldots,g_k)$, and conclude that the i-th syzygy module $\mathrm{syz}_i(M)$ is well-defined up to isomorphism.

Problem 3: Let R be a Noetherian ring and $M = \langle f_1,\ldots,f_k\rangle = \langle g_1,\ldots,g_l\rangle \leq R^r$.

(a) Show that the dotted maps in the following diagramm exists such that it commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{syz}(f_1,\ldots,f_k) & \xrightarrow{i_f} & R^k & \xrightarrow[e_i \mapsto f_i]{\varphi_f} & M & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle \nu} & & \Big\| & & \\
0 & \longrightarrow & \mathrm{syz}(g_1,\ldots,g_l) & \xrightarrow{i_g} & R^l & \xrightarrow[e_i \mapsto g_i]{\varphi_g} & M & \longrightarrow & 0
\end{array}$$

(b) Prove that its *total complex* is exact with $\mathrm{Im}(\begin{pmatrix}\varphi_f & 0\\ -\nu & i_g\end{pmatrix}) \cong R^l \cong \mathrm{Ker}((\mathrm{id}_M,\varphi_g))$

$$0 \longrightarrow \mathrm{syz}(f_1,\ldots,f_k) \xrightarrow{\begin{pmatrix} i_f \\ \mu\end{pmatrix}} R^k \oplus \mathrm{syz}(g_1,\ldots,g_l) \xrightarrow{\begin{pmatrix}\varphi_f & 0\\ -\nu & i_g\end{pmatrix}} M \oplus R^l \xrightarrow{(\mathrm{id}_M,\varphi_g)} M \longrightarrow 0$$

(c) Show that $\mathrm{syz}(g_1,\ldots,g_l) \oplus R^k \cong R^l \oplus \mathrm{syz}(f_1,\ldots,f_k)$.
Hint: Use a suitable split exact sequence.

Problem 4: Write a Singular procedure to compute the ecart of a given polynomial and use this to implement a normal form algorithm for non-global orderings.