TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

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Algebraic Geometry

Summer Semester 2015 - Problem Set 5

Due May 22, 2013, 11:00 am

In all exercises, the ground field k is assumed to be algebraically closed.

Problem 1.

(a) Let $X \subset \mathbb{A}^n$ an affine variety, and let $P \in X$ be a point. Show that $\mathscr{O}_{X,P} \cong \mathscr{O}_{\mathbb{A}^n,P}/I(X)\mathscr{O}_{\mathbb{A}^n,P}$, where $I(X)\mathscr{O}_{\mathbb{A}^n,P}$ denotes the ideal in $\mathscr{O}_{\mathbb{A}^n,P}$ generated by all quotients $\frac{f}{1}$ for $f \in I(X)$.

(b) Let $\mathscr{F}$ be a sheaf on a topological space X. Let $P \in X$ and $P \in U \subset X$ an open neighbourhood of P. Show that $\mathscr{F}_P \cong (\mathscr{F}|_U)_P$.

Problem 2. Let X be a topological space and $\mathscr{F}$ be a sheaf on X. Let $s \in \mathscr{F}(U)$ and denote by $\operatorname{Supp}(s) = \{P \in U \mid s_P \neq 0\}$ the *support of* s. Show that $\operatorname{Supp}(s)$ is a closed subset of U. We define the *support of* $\mathscr{F}$ to be $\{P \in X \mid \mathscr{F}_p \neq 0\}$. Is $\operatorname{Supp}(\mathscr{F})$ always closed?

Problem 3. In this problem we want to consider operations on sheaves, which give us new (pre-)sheaves.

(a) Let X, Y topological spaces and let $\mathscr{F}$ be a sheaf on X. Let $f : X \to Y$ be a continuous map. We define the *direct image sheaf* $f_*\mathscr{F}$ to be $f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$ for an open subset $V \subset Y$.

(b) Let U be an open subset of a topological space X. Let $\mathscr{F}$ be a sheaf on U. Let $j : U \to X$ be the natural inclusion. For $V \subset X$ open define $j_!\mathscr{F}(V) = \mathscr{F}(V)$ if $V \subset U$ and $j_!\mathscr{F}(V) = 0$ otherwise.

Show that the constructions above give new presheaves and compute their stalks. Are they always sheaves?

Problem 4. Let $\mathscr{F}$ be any sheaf on X. Define the *sheaf of discontinuous sections* $\mathscr{F}_{disc}$ of $\mathscr{F}$ as follows. For each open set $U \subset X$ we let $\mathscr{F}_{disc}(U) = \{s : U \to \bigcup_{P \in U} \mathscr{F}_P \mid s(P) \in \mathscr{F}_P \text{ for all } P \in U\}$. Show that $\mathscr{F}(U) \subset \mathscr{F}_{disc}(U)$, for all $U \subset X$ open, in a natural way. Prove that $\mathscr{F}_{disc}$ is a sheaf on X and $\mathscr{F}_P \subset (\mathscr{F}_{disc})_P$ for all $P \in X$.