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Algebraic Geometry

Summer Semester 2015 - Problem Set 12

Due July 10, 2015, 11:00 am

In all exercises, the ground field k is assumed to be algebraically closed.

Problem 1. Let $X \subset \mathbb{P}^3$ the degree-3 Veronese embedding of $\mathbb{P}^1$. Since X is isomorphic to $\mathbb{P}^1$ we know that X is necessarily a smooth curve. Verify this directly with the projective Jacobian criterion.

Problem 2. Let $X \subset \mathbb{P}^N$ be a projective variety of dimension n. Show that:

(a) There exists an injective morphism from X to $\mathbb{P}^{2n+1}$. (*Hint:* The *secant variety* $\operatorname{Sec}(X)$ of X is the closure of the set of all points of $\mathbb{P}^N$ contained in a line defined by two points of X. Show that $\dim \operatorname{Sec}(X) \leq 2n+1$.)

(b) There is in general no such morphism that is an isomorphism onto its image.

Problem 3. Let $n \geq 2$. Prove:

(a) Every smooth hypersurface in $\mathbb{P}^n$ is irreducible.

(b) A general hypersurface in $\mathbb{P}^n_{\mathbb{C}}$ is smooth. More precisely, for $d > 0$ the vector space $\mathbb{C}[x_0, \ldots, x_n]_d$ has dimension $\binom{n+d}{n}$ and so the space of homogeneous degree-d polynomials modulo scalars can be identified with $\mathbb{P}_{\mathbb{C}}^{\binom{n+d}{d}-1}$. Show that the subset of this projective space of all (classes of) irreducible polynomials f such that $V_p(f)$ is smooth is dense and open.

Problem 4. Let $\operatorname{char}(k) \neq 2$ and let $f \in k[x_0, x_1, x_2]$ be a homogeneous polynomial whose partial derivatives do not vanish simultaneously at any point of $X = V_p(f)$. Consider the morphism $F : X \to \mathbb{P}^2$, with $F(a) = (\frac{\partial f}{\partial x_0}(a) : \frac{\partial f}{\partial x_1}(a) : \frac{\partial f}{\partial x_2}(a))$ for $a \in X$. The projective variety $F(X)$ is called the *dual curve* to X.

(a) Find a geometric description of F. What does it mean geometrically if $F(a) = F(b)$ for two distinct points $a, b \in X$?

(b) If X is a conic, prove that $F(X)$ is also a conic.