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# Algebraic Geometry

Summer Semester 2015 - Problem Set 11

Due July 3, 2015, 11:00 am

In all exercises, the ground field $k$ is assumed to be algebraically closed.

**Problem 1.** Let $X = V(x_2^2 - x_1^2 - x_1^3) \subset \mathbb{A}^2$. We know that $\mathbb{A}^1$ is not isomorphic to $X$. Show that $\widetilde{X}$ is isomorphic to $\mathbb{A}^1$. Can you interpret these results geometrically?

**Problem 2.** Let $\widetilde{\mathbb{A}^3}$ be the blow-up of $\mathbb{A}^3$ at the line $V(x_1, x_2) \cong \mathbb{A}^1$. Show that its exceptional set is isomorphic to $\mathbb{A}^1 \times \mathbb{P}^1$. When do the strict transforms of two lines in $\mathbb{A}^3$ through $V(x_1, x_2)$ intersect in the blow-up? Describe the geometric meaning of the points in the exceptional set in the context of Example 9.15.

**Problem 3.** Let $X \subset \mathbb{A}^n$ be an affine vairety, and let $Y_1, Y_2 \subsetneq X$ be irreducible, closed subsets, no-one contained in the other. Let $\tilde{X}$ be the blow-up of $X$ at the ideal $I(Y_1) + I(Y_2)$. Show that the strict transforms of $Y_1$ and $Y_2$ in $\tilde{X}$ are disjoint.

**Problem 4.** Let $I \trianglelefteq k[x_1, \dots, x_n]$ be an ideal, and assume that the corresponding affine variety $X = V(I) \subset \mathbb{A}^n$ contains the origin. Consider the blow-up $\widetilde{X} \subset \widetilde{\mathbb{A}^n} \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$. at $x_1, \dots, x_n$ and denote the homogeneous coordinates of $\mathbb{P}^{n-1}$ by $y_1, \dots, y_n$.

By Example 9.15 we know that $\widetilde{\mathbb{A}^n}$ can be covered by affine spaces, with one coordinate patch being $\mathbb{A}^n \to \widetilde{\mathbb{A}^n} \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$, where $(x_1, y_2, \dots, y_n) \mapsto ((x_1, x_1 y_2, \dots, x_1 y_n), (1 : y_2 : \dots : y_n))$.

Prove that on this coordinate patch the blow-up $\widetilde{X}$ is given as the zero locus of the polynomials $\frac{f(x_1, x_1 y_2, \dots, x_1 y_n)}{x_1^{\mathrm{mindeg}(f)}}$, for all non-zero $f \in I$, where $\mathrm{mindeg}(f)$ denotes the smallest degree of a monomial in $f$.