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Computer Algebra

Winter Semester 2014 - Problem Set 6

Due December 11, 2014, 10:00

Problem 1: Let $A = \mathbb{Q}[x,y]_{\langle x,y\rangle}/\langle xy\rangle$. Compute the Betti numbers of the A-module $M = \langle (x^2,y),(x,y)\rangle \leq A^2$ by hand. Use Singular to check your results.

Problem 2: Let R be a local Noetherian ring, let M be a finitely generated R-module, and let $\{f_1,\dots,f_k\}$, $\{g_1,\dots,g_k\}$ be two minimal sets of generators. Prove that $\operatorname{syz}(f_1,\dots,f_k) \cong \operatorname{syz}(g_1,\dots,g_k)$, and conclude that the i-th syzygy module $\operatorname{syz}_i(M)$ is well-defined up to isomorphism.

Problem 3: Let R be a Noetherian ring and $M = \langle f_1,\dots,f_k\rangle = \langle g_1,\dots,g_l\rangle \leq R^r$.

(a) Show that the dotted maps in the following diagramm exists such that it commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{syz}(f_1,\dots,f_k) & \xrightarrow{i_f} & R^k & \xrightarrow[e_i\mapsto f_i]{\varphi_f} & M & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle\mu} & & \Big\downarrow{\scriptstyle\nu} & & \Big\| & & \\
0 & \longrightarrow & \operatorname{syz}(g_1,\dots,g_l) & \xrightarrow{i_g} & R^l & \xrightarrow[e_i\mapsto g_i]{\varphi_g} & M & \longrightarrow & 0
\end{array}$$

(b) Prove that its *total complex* is exact with $\operatorname{Im}\left(\begin{pmatrix}\varphi_f & 0\\ -\nu & i_g\end{pmatrix}\right) \cong R^l \cong \operatorname{Ker}((\operatorname{id}_M,\varphi_g))$

$$0 \longrightarrow \operatorname{syz}(f_1,\dots,f_k) \xrightarrow{\begin{pmatrix} i_f\\ \mu\end{pmatrix}} R^k \oplus \operatorname{syz}(g_1,\dots,g_l) \xrightarrow{\begin{pmatrix}\varphi_f & 0\\ -\nu & i_g\end{pmatrix}} M \oplus R^l \xrightarrow{(\operatorname{id}_M,\varphi_g)} M \longrightarrow 0$$

(c) Show that $\operatorname{syz}(g_1,\dots,g_l)\oplus R^k \cong R^l \oplus \operatorname{syz}(f_1,\dots,f_k)$.
Hint: Use a suitable split exact sequence.

Problem 4: Change your Singular procedure computing a Gröbner basis in such a way that

1. the pair set P is sorted in ascending order w.r.t. $\operatorname{lcm}(\mathrm{LM}_>(f_1),\mathrm{LM}_>(f_2))$, $f_1,f_2\in P$.
2. it takes an optional paramter such that if this optional parameter is the string "minimal", the procedure returns a minimal Gröbner basis, and if this optional parameter is missing, the procedure just returns some standard basis as before.

Hint: If you add `list #` at the end of the input of your procedure, then your procedure allows for any number of optional parameters. You can test with `size(#)` the number of optional parameters the user has provided, while `#[i]` gives you the i-th optional parameter.