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Computer Algebra

Winter Semester 2014 - Problem Set 1

Due November 6, 2013, 10:00 am

Problem 1. Let γ be a totally ordered monoid and R be a γ-filtered ring.

(a) Let $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ be a short exact sequence of R-modules and $F_\bullet$ a γ-filtration on M. Define by $F_pM' := \varphi^{-1}F_pM$ and $F_pM'' := \psi F_pM$ filtrations $F_\bullet$ on M' and M''. Show that the above sequence is strict with respect to $F_\bullet$ (i.e. the maps in the above seqeunce are strict with respect to $F_\bullet$).

(b) Let $0 \to M \xrightarrow{\varphi} N \xrightarrow{\psi} P \to 0$ be a short exact sequence. Assume that M, N and P are $\mathbb{Z}$-filtered modules with filtrations $F^X_\bullet$ such that $F^X_p X = 0$ for $p < 0$ and $F^X_1 X = X$ (where $X \in \{M, N, P\}$). Show that the following are equivalent:

(i) The above sequence is strict.

(ii) The above sequence induces short exact sequences

$$0 \to F^M_0 M \to F^N_0 N \to F^P_0 P \to 0$$

and

$$0 \to \mathrm{gr}^{F^M}_1 M \to \mathrm{gr}^{F^N}_1 N \to \mathrm{gr}^{F^P}_1 P \to 0.$$

(c) Let $\varphi : M \to N$ be a γ-filtered homomorphism, where M and N are γ-filtered R-modules with filtrations $F_\bullet$ and $G_\bullet$, respectively. Under a suitable hypothesis on $F_\bullet$ and $G_\bullet$ show that the following are equivalent:

(i) φ is strict with respect to $F_\bullet$ and $G_\bullet$,

(ii) $\mathrm{gr}^F_\bullet\, \varphi(M) = \mathrm{gr}^G_\bullet\, \varphi(M)$.

Note that there is a map $F_\bullet\varphi(M) \xrightarrow{\mathrm{id}} G_\bullet\varphi(M)$.

Problem 2. Monomial orderings arise in a variety of ways. One possibility is to use matrices to define monomial orderings: The matrix $A \in \mathrm{GL}(n, \mathbb{R})$ defines a monomial ordering $>_A$ on $\mathrm{Mon}(x_1, \ldots, x_n)$ by setting

$$x^\alpha >_A x^\beta :\Leftrightarrow A\alpha > A\beta,$$

where $>$ on the right–hand side is the lexicographical ordering on $\mathbb{R}^n$.

One can also define new monomial orderings from "known" orderings using so-called product orderings: Consider a monomial ordering $>_1$ on $\mathrm{Mon}(x_1, \ldots, x_{n_1})$ and a monomial ordering $>_2$ on $\mathrm{Mon}(y_1, \ldots, y_{n_2})$. Then the product ordering or block ordering $>$, also denoted by $(>_1, >_2)$, on $\mathrm{Mon}(x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2})$, is defined as

$$x^\alpha y^\beta > x^{\alpha'} y^{\beta'} :\Leftrightarrow x^\alpha >_1 x^{\alpha'} \text{ or } \left(x^\alpha = x^{\alpha'} \text{ and } y^\beta >_2 y^{\beta'} \right).$$

Given a vector $w = (w_1, \ldots, x_n)$ of integers, we define the weighted degree of x^α by

$$w\text{-}\deg(x^\alpha) := \langle w, \alpha \rangle := w_1\alpha_1 + \cdots + w_n\alpha_n,$$

that is, the variable x_i has degree w_i. For a polynomial $f = \sum_\alpha a_\alpha x^\alpha$, we define the weighted degree,

$$w\text{-}\deg(f) := \max\{w\text{-}\deg(x^\alpha) \mid a_\alpha \neq 0\}.$$

Using the weighted degree in the definition of $>_{dp}$, respectively $>_{ds}$ (cf. Example 1.2.8 in the SINGULAR book by Greuel, Pfister), with all $w_i > 0$, instead of the usual degree, we obtain the weighted reverse lexicographical ordering $>_{wp(w_1,\ldots,w_n)}$, respectively the negative weighted reverse lexicographical ordering $>_{ws(w_1,\ldots,w_n)}$.

(a) Show that $>_A$ is indeed a monomial ordering on $\mathrm{Mon}(x_1, \ldots, x_n)$.

(b) Determine matrices $A \in \mathrm{GL}(n, \mathbb{R})$ defining the orderings

  (i) $>_{ws(5,3,4)}$ on $\mathrm{Mon}(x_1, x_2, x_3)$ with $n = 3$,

  (ii) $(>_{dp}, >_{ls})$ on $\mathrm{Mon}(x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2})$ with $n = n_1 + n_2$,

  (iii) $(>_{ds}, >_{wp(7,1,9)})$ on $\mathrm{Mon}(x_1, \ldots, x_{n_1}, y_1, y_2, y_3)$ with $n = n_1 + 3$.

Problem 3. Write a SINGULAR procedure, having a list $P = ((g_1, h_1), \ldots, (g_r, h_r))$ of pairs of polynomials, an ideal $I = \langle f_1, \ldots, f_s \rangle$ and a polynomial f as input and returning the extended pair set $P = P \cup ((f, f_1), \ldots, (f, f_s))$ as output.

Don't forget to add at least one example to your procedure.