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# Computer Algebra

Winter Semester 2013 - Problem Set 5

Due December 3, 2013, 12:00

**Problem 1:** Let $A = \mathbb{Q}[x,y]_{\langle x,y\rangle}/\langle xy\rangle$. Compute the Betti numbers of the $A$-module $M = \langle (x^2,y),(x,y)\rangle \leq A^2$.

**Problem 2:** Let $R$ be a ring and let $M, N$ be two $R$-modules. Show that any exact sequence $0 \to M \overset{\varphi}{\to} N \overset{\psi}{\to} R^r \to 0$ splits, that is, there exists an isomorphism $\lambda : M \oplus R^r \to N$ such that the following diagram commutes (i.e. $\lambda \circ i = \varphi$, $\psi \circ \lambda = \pi$):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\ \varphi\ } & N & \xrightarrow{\ \psi\ } & R^r & \longrightarrow & 0 \\
 & & \uparrow{=} & & \uparrow{\lambda} & & \uparrow{=} & & \\
0 & \longrightarrow & M & \xrightarrow{\ i\ } & M \oplus R^r & \xrightarrow{\ \pi\ } & R^r & \longrightarrow & 0
\end{array}$$

**Problem 3:** Let $R$ be a Noetherian ring and $M = \langle f_1,\dots,f_k\rangle = \langle g_1,\dots,g_l\rangle \leq R^r$.

(a) Show that the dotted maps in the following diagramm exists such that it commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{syz}(f_1,\dots,f_k) & \xrightarrow{\ i_f\ } & R^k & \xrightarrow[e_i \mapsto f_i]{\varphi_f} & M & \longrightarrow & 0 \\
 & & \downarrow{\mu} & & \downarrow{\nu} & & \downarrow{=} & & \\
0 & \longrightarrow & \operatorname{syz}(g_1,\dots,g_l) & \xrightarrow{\ i_g\ } & R^l & \xrightarrow[e_i \mapsto g_i]{\varphi_g} & M & \longrightarrow & 0
\end{array}$$

(b) Prove that its *total complex* is exact with $\operatorname{Im}(\begin{pmatrix} i_g & -\nu \\ 0 & \varphi_f \end{pmatrix}) \cong R^l \cong \operatorname{Ker}((\varphi_g, \mathrm{id}_M))$

$$0 \longrightarrow \operatorname{syz}(f_1,\dots,f_k) \xrightarrow{\begin{pmatrix} \mu \\ i_f \end{pmatrix}} \operatorname{syz}(g_1,\dots,g_l) \oplus R^k \xrightarrow{\begin{pmatrix} i_g & -\nu \\ 0 & \varphi_f \end{pmatrix}} R^l \oplus M \xrightarrow{(\varphi_g,\mathrm{id}_M)} M \longrightarrow 0$$

(c) Use Problem 2 to show that $\operatorname{syz}(g_1,\dots,g_l) \oplus R^k \cong R^l \oplus \operatorname{syz}(f_1,\dots,f_k)$.

(d) Show that, if $k = l$, then $\operatorname{syz}(f_1,\dots,f_k) \cong \operatorname{syz}(g_1,\dots,g_l)$, and conclude that, if $R$ is local, $\operatorname{syz}_k(M)$ is well-defined up to isomorphism for any finitely generated $M$.

**Problem 4:** Change your Singular procedure computing a Gröbner basis in such a way that

1. the pair set $P$ is sorted in ascending order w.r.t. $\operatorname{lcm}(\mathrm{LM}_>(f_1), \mathrm{LM}_>(f_2))$, $f_1, f_2 \in P$.
2. it takes an optional paramter such that if this optional parameter is the string “minimal”, the procedure returns a minimal Gröbner basis, and if this optional parameter is missing, the procedure just returns some standard basis as before.

Hint: If you add `list #` at the end of the input of your procedure, then your procedure allows for any number of optional parameters. You can test with `size(#)` the number of optional parameters the user has provided, while `#[i]` gives you the $i$-th optional parameter.