TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

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Computer Algebra

Winter Semester 2013 - Problem Set 3

Due November 19, 2013, 12:00

Problem 1: Check by hand whether the following inclusions are correct:

(a) $xy^3 - z^2 + y^5 - z^3 \in \langle -x^3 + y, x^2y - z \rangle \trianglelefteq \mathbb{Q}[x, y, z]$

(b) $x^3z - 2y^2 \in \langle yz - y, xy + 2z^2, y - z \rangle \trianglelefteq \mathbb{Q}[x, y, z]$

(c) $x^3z - 2y^2 \in \langle yz - y, xy + 2z^2, y - z \rangle \trianglelefteq \mathbb{Q}[x, y, z]_{\langle x,y,z \rangle}$

Problem 2: (*Product Criterion*) Let $>$ be a global monomial ordering on $\mathrm{Mon}(x_1, \dots, x_n)$. Let $f, g \in K[x_1, \dots, x_n]$ be polynomials such that $\mathrm{lcm}(\mathrm{LM}_>(f), \mathrm{LM}_>(g)) = \mathrm{LM}_>(f) \cdot \mathrm{LM}_>(g)$. Prove that

$$\mathrm{NF}(\mathrm{spoly}(f, g) \mid \{f, g\}) = 0.$$

Hint: Assume that $\mathrm{LC}_>(f) = \mathrm{LC}_>(g) = 1$ and claim that $\mathrm{spoly}(f, g) = -\operatorname{tail}(g) \cdot f + \operatorname{tail}(f) \cdot g$ is a standard representation.

Problem 3: Let $R := K[x_1, \dots, x_n]$ be a polynomial ring, $>$ a global monomial ordering on $\mathrm{Mon}(x_1, \dots, x_n)$, and $I \trianglelefteq R$ an ideal. Let $G = \{g_1, \dots, g_k\}$ be a generating set of I. Consider the complex

$$C: \quad \widetilde{G}_\bullet R^k \longrightarrow F_\bullet^> R \longrightarrow F_\bullet^>(R/I) \longrightarrow 0, \qquad e_i \longmapsto g_i$$

in which the filtration $F_\bullet^>$ on R is induced by the monomial ordering $>$, the filtration $F_\bullet^>$ on R/I is induced by the filtration $F_\bullet^>$ on R and the quotient map $R \to R/I$, and the filtration $\widetilde{G}_\bullet$ on R^k denotes the coarse Schreyer filtration on R^r induced by $g_1, \dots, g_k$. Show that G is a standard basis with respect to $>$ if the sequence

$$\operatorname{gr} C: \quad \operatorname{gr}^{\widetilde{G}} R^k \longrightarrow \operatorname{gr}^{F^>} R \longrightarrow \operatorname{gr}^{F^>} R/I \longrightarrow 0$$

is exact.

Problem 4: Write a SINGULAR procedure to compute the reduced normal form of a given polynomial $f \in K[x_1, \dots, x_n]$ with respect to a given finite list of polynomials $G \subseteq K[x_1, \dots, x_n]$ and a global monomial ordering $>$ without the use of the commands `reduce` and `NF`.