HOMEWORK 7

Define your terminology and explain notation. If you require a standard result, such as one of the Sylow theorems, then state it (or cite a result from your textbook) before you use it; otherwise give clear and complete proofs of your claims. The problems are of equal value, 6 points each. Partial solutions will be considered on their merits.

Exercise 1. Let E be a module over a commutative ring R, $S \subseteq R$ a multiplicatively closed subset, and $I \leq R$ an ideal.

(a) Show that $S^{-1}E \cong S^{-1}R \otimes_R E$ is a natural isomorphism.

(b) Show that $E/IE \cong (R/I) \otimes_R E$ is a natural isomorphism.

(c) Compute $(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z})$.

Exercise 2. Let S be a commutative R-algebra.

(a) Show that $S \otimes_R N$ is a flat S-module if N is a flat R-module.

(b) Show that M is a flat R-module if M is a flat S-module and S is a flat R-algebra.

(c) Show that $M \otimes_R M'$ is R-flat if both M and M' are R-flat.

Exercise 3. A flat module M is called *faithfully flat* if $M \otimes N = 0$ implies $N = 0$ for all modules N. Prove that each of the following conditions is equivalent to faithful flatness:

(a) M is flat and $u \otimes \mathrm{id}_M = 0$ implies $u = 0$ for all $u\colon N \to N'$.

(b) M is flat and $\mathfrak{m}M \neq M$ for all maximal ideals $\mathfrak{m} \leq R$.

(c) $N' \to N \to N''$ is exact if and only if $N' \otimes M \to N \otimes M \to N'' \otimes M$ is exact.

Exercise 4. Finish the proof of the "naturality" part of Lang XVI.Ex.11 (finitely presented case) following the outline given in class and using what we learned about homotopy.