HOMEWORK 6

Define your terminology and explain notation. If you require a standard result, such as one of the Sylow theorems, then state it (or cite a result from your textbook) before you use it; otherwise give clear and complete proofs of your claims. The problems are of equal value, 6 points each. Partial solutions will be considered on their merits.

Exercise 1 (Zariski topology)**.** Let A be a commutative ring and $\operatorname{Spec} A$ the set of its proper prime ideals. For any subset $S \subseteq A$, set $V(S) := \{\mathfrak{p} \in \operatorname{Spec} A \mid S \subseteq \mathfrak{p}\}$ and $D(S) := \operatorname{Spec} A \setminus V(S)$. Note that $V(S) = V(I)$ where $I = \langle S \rangle$ is the ideal generated by S.

A topological space X is called *irreducible* if $X = X_1 \cup X_2$ with $X_i \subset X$ closed implies that $X = X_i$ for some i. A subset $Y \subset X$ is called *irreducible*, if it is irreducible with its relative topology ($U \subseteq Y$ is open if and only if $U = Y \cap U'$ with $U' \subseteq X$ open).

1. Show that the $V(S)$ (resp. $D(S)$) are the closed (resp. open) sets of a topology on $\operatorname{Spec} A$, the so-called *Zariski topology*.
2. Show that the $D(f) := D(\{f\})$, $f \in A$, form a basis of the Zariski topology.
3. Interpret $I \mapsto V(I)$ as an inclusion reversing bijection.
4. Show that a point $\mathfrak{p} \in \operatorname{Spec} A$ is closed if and only if $\mathfrak{p} \leq A$ is a maximal ideal.
5. Show that a radical ideal $I \leq A$ is prime if and only if $V(I)$ is irreducible.
6. Describe $\operatorname{Spec} K$ for K a field. Sketch $\operatorname{Spec} \mathbb{Z}$, where is $\mathfrak{p} = \langle 0 \rangle$?
7. Show that Spec is a contravariant functor from rings to topological spaces.
8. For $f \in A$, show that $\operatorname{Spec} A_f$ is homeomorphic to the open subset $D(f)$ of $\operatorname{Spec} A$.

Exercise 2. Let B be a commutative A-algebra, E an A-module, and F a B-module.

1. Explain why F is an A-module, and why $B \otimes_A E$ is a B-module.
2. Show that $\operatorname{Hom}_B(B \otimes_A E, F) \cong \operatorname{Hom}_A(E, F)$.

Exercise 3. Show that tensor products commute with direct limits.