HOMEWORK 4

Define your terminology and explain notation. If you require a standard result, such as one of the Sylow theorems, then state it (or cite a result from your textbook) before you use it; otherwise give clear and complete proofs of your claims. The problems are of equal value, 6 points each. Partial solutions will be considered on their merits.

Exercise 1. Let $\alpha = \sqrt{2+\sqrt{2}} \in \mathbb{R}$.

(a) Determine the minimal polynomial f of α over $\mathbb{Q}$.

(b) Show that f splits over $E = \mathbb{Q}(\alpha)$.

(c) Show that $\mathrm{Gal}(E/\mathbb{Q})$ contains an element of order 4.

Exercise 2. Compute the Galois group of the (splitting field of the) polynomial $(x^3-2)(x^2-5)$ over $\mathbb{Q}$.

Exercise 3. Let $\zeta = e^{2\pi i/17}$, $F = \mathbb{Q}(\zeta)$, and $K = \mathbb{Q}(\zeta+\zeta^{-1})$.

(a) Show that $K/\mathbb{Q}$ and $F/\mathbb{Q}$ are Galois extensions and determine their Galois groups.

(b) Show that any $F \to \mathbb{C}$ maps K to $\mathbb{R}$.

Exercise 4. Let K be a finite non-normal extension of $\mathbb{Q}$ and suppose that K has no subfield other than $\mathbb{Q}$ and itself. Show that the identity is the only automorphism of K.

Exercise 5. Let F be a field and E a Galois extension of F if finite degree. Let $h(x) \in F[X]$ be an irreducible polynomial. Show that all irreducible factors of $h(x)$ in $E[X]$ have the same degree. Give an example to show that this need not be true if E is merely a finite extension.

Exercise 6. Find the Galois group of $x^{12}-3$ over $\mathbb{Q}$.