HOMEWORK 1

Define your terminology and explain notation. If you require a standard result, such as one of the Sylow theorems, then state it (or cite a result from your textbook) before you use it; otherwise give clear and complete proofs of your claims. The problems are of equal value, 6 points each. Partial solutions will be considered on their merits.

Exercise 1.

(a) Show that for $a, b \in \mathbb{Q}$ with $\sqrt{a} + \sqrt{b} \neq 0$, we have $\mathbb{Q}(\sqrt{a} + \sqrt{b}) = \mathbb{Q}(\sqrt{a}, \sqrt{b})$.

(b) Show that $\mathbb{Q}(\sqrt{2} + \sqrt{3})/\mathbb{Q}$ is algebraic of degree 4.

Exercise 2. Let $E = \mathbb{Q}(\alpha)$ where α is a root of the equation $\alpha^3 + \alpha^2 + \alpha + 2 = 0$.

(a) What is the minimal polynomial of α over $\mathbb{Q}$.

(b) Express $(\alpha - 1)^{-1}$ in terms of the $\mathbb{Q}$-basis 1,α,α^2 of E.

Exercise 3.

(a) If both a and b are algebraic over F, show that $a + b$ is algebraic over F.

(b) Let $F(\alpha)/F$ be an algebraic extension of odd degree. Show that $F(\alpha) = F(\alpha^2)$.

Exercise 4.

(a) Let E, F be two extensions of K. Show that $[EF : K] \leq [E : K][F : K]$.

(b) Prove that equality holds in (a) if $[E : K]$ and $[F : K]$ are coprime.

Exercise 5. Compute the splitting fields and the degree over $\mathbb{Q}$.

(a) $(X^3 - 2)(x^2 - 2)$

(b) $X^5 - 7$

Exercise 6. Let $f \in F[X]$ be a polynomial of degree n with splitting field K. Show that $[K : F]$ divides $n!$.