First Name:______________________ **Last Name**:______________________

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OKLAHOMA STATE UNIVERSITY
Department of Mathematics

MATH 2144 (Calculus I)
Instructor: Dr. Mathias Schulze

MIDTERM 2
March 29, 2011

Duration: 50 minutes

No aids allowed.

This examination paper consists of **5** pages and **8** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer **all** questions.

To obtain credit, you must give arguments to support your answers.

For graders' use:

	Score
1 (5)	
2 (5)	
3 (5)	
4 (5)	
5 (5)	
6 (5)	
7 (5)	
8 (5)	
Total (40)	

1. [5] For $f(x) = xe^{-x}$, compute $f'(x)$ and $f''(x)$. Then find the 1000th derivative.

   Solution: Since $((x-k)e^{-x})' = -(x-(k+1))e^{-x}$, we have $f^{(k)} = (-1)^k(x-k)e^{-x}$, and hence $f^{(1000)} = (x-1000)e^{-x}$.

2. [5] Use implicit differentiation to find the equation of the tangent line to the curve $x^2 + xy + y^2 = 3$ at the point $(1, 1)$.

   Solution: Differentiation gives

$$2x + y + xy' + 2yy' = 0.$$

   Setting $m = y'(1, 1)$ and substituting $(x, y) = (1, 1)$, this becomes

$$2 + 1 + m + 2m = 0.$$

   So $m = -1$ and the equation of the tangent line at $(1, 1)$ reads

$$y = 1 + m(x-1) = -x + 2.$$

3. [5] Differentiate:

   (a) $y = \log_5(xe^x)$

   (b) $y = x^{\sin x}$ (logarithmic differentiation!)

Solution:

(a)

$$y' = \frac{(x+1)e^x}{\ln(5)xe^x} = \left(1 + \frac{1}{x}\right)\ln\frac{1}{5}$$

(b) Applying ln gives

$$\ln y = \sin(x)\ln(x).$$

Then differentiate to obtain

$$\frac{y'}{y} = \cos(x)\ln(x) + \frac{\sin x}{x}$$

which yields

$$y' = x^{\sin(x)}\left(\cos(x)\ln(x) + \frac{\sin x}{x}\right).$$

4. [5] If a snow ball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

Solution: The surface area of the snow ball is

$$A = 4\pi r^2 = \pi d^2$$

where $r = r(t)$ and $d = d(t)$ denote the radius and diameter, respectively. So, at the time $t = t_0$ when $d(t_0) = 10$, we have

$$-1 = A'(t_0) = 2\pi d(t_0)d'(t_0) = 20\pi d'(t_0)$$

and hence $d'(t_0) = -\frac{1}{20\pi}$.

5. [5] The radius of a circular disc is given as 10 cm with a maximum error in measurement of 0.1 cm. Use differentials to estimate the relative error in the calculated area of the disc.

Solution: From $A = \pi r^2$, we compute $dA = 2\pi r dr$, and then the relative error is

$$\frac{\Delta A}{A} \approx \frac{dA}{A} = 2\frac{dr}{r} = 0.02 = 2\%.$$

6. [5] Find the absolute maximum and minimum values of the function $f(x) = \frac{x^2-4}{x^2+4}$, where $-4 \leq x \leq 4$.

Solution: Since

$$f'(x) = \frac{16x}{(x^2+4)^2},$$

the critical numbers are $-4, 0, 4$. Therefore,

$$f(\pm 4) = \frac{3}{5} \text{ and } f(0) = -1$$

are the maximum and minimum values.

7. [5] Compute the limits.

(a) $\lim_{x\to 1} \frac{x^9-1}{x^5-1}$

(b) $\lim_{x\to -\infty} x^2 e^x$

Solution:

(a) $\lim_{x\to 1} \frac{x^9-1}{x^5-1} = \lim_{x\to 1} \frac{9x^8}{5x^4} = \lim_{x\to 1} \frac{9x^4}{5} = \frac{9}{5}$

(b)

$$\begin{aligned}\lim_{x\to -\infty} x^2 e^x &= \lim_{x\to -\infty} \frac{x^2}{e^{-x}} = -\lim_{x\to -\infty} \frac{2x}{e^{-x}} \\ &= \lim_{x\to -\infty} \frac{2}{e^{-x}} = 2 \lim_{x\to -\infty} e^x = 0\end{aligned}$$

8. [5] A box with a square base and open top must have a volume of $32,000\,\mathrm{cm}^3$. Find the dimensions of the box that minimize the amount of material used.

Solution: Let x be the base width and h be the height of the box. Then the amount of material used is proportional to the surface area $A = x^2 + 4xh$, while the volume equals $32000 = V = x^2 h$. Using the latter to eliminate h, gives $A(x) = x^2 + \frac{4V}{x}$. Then, setting $A'(x) = 2x - \frac{4V}{x^2} = 0$, yields $x = \sqrt[3]{2V} = 40\,\mathrm{cm}$ and $h = \frac{V}{x^2} = 20\,\mathrm{cm}$.

End of examination
Total pages: 5
Total marks: 40