HOMEWORK 9

Define your terminology and explain notation. If you require a standard result, such as one of the Sylow theorems, then state it (or cite a result from your textbook) before you use it; otherwise give clear and complete proofs of your claims. The problems are of equal value, 6 points each. Partial solutions will be considered on their merits.

Exercise 1.

1. A commutative ring with unique maximal ideal is called *local*. Show that $A_{\mathfrak{p}}$ is local for any prime ideal $\mathfrak{p} \leq A$.
2. Let $\phi\colon A \to B$ be a homomorphism of commutative rings, let $\mathfrak{q} \leq B$ be a prime ideal and $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$. Construct a canonical ring homomorphism $\phi_{\mathfrak{q}}\colon A_{\mathfrak{p}} \to B_{\mathfrak{q}}$.
3. Let $f \in A$. Identify A_f with $A[X]/\langle Xf - 1\rangle$.

Exercise 2. Let $\mathfrak{a} \leq A$ be an ideal and $S \subset A$ a multiplicatively closed subset. Denote by $S/\mathfrak{a}$ the image of S under the canonical map $A \to A/\mathfrak{a}$. Show that $(S/\mathfrak{a})^{-1}(A/\mathfrak{a}) \cong (S^{-1}A)/((S^{-1}A)\cdot\mathfrak{a})$.

Exercise 3. Let A be a commutative ring. Define the localization of an A-module. (A good definition comes with a universal property.)

Exercise 4. Show that localization, as defined in the preceding exercise, is an exact functor.

Exercise 5.

1. Let A be a PID. Show that a nonzero ideal in A is maximal if and only if it is a prime ideal.
2. Let A be an integral domain. Show that A is a unique factorization domain if and only if every nonzero prime ideal contains a nonzero principal prime ideal.

Exercise 6. Show that $K[X^2, X^3]$ is not factorial.

Exercise 7. Show that $\mathbb{Z}[\sqrt{-5}]$ is not a PID and not a UFD.

Exercise 8. Show that $\mathbb{Z}[i]$ is a Euclidean domain and find the factorization of 13 into irreducible factors.