HOMEWORK 8

Define your terminology and explain notation. If you require a standard result, such as one of the Sylow theorems, then state it (or cite a result from your textbook) before you use it; otherwise give clear and complete proofs of your claims. The problems are of equal value, 6 points each. Partial solutions will be considered on their merits.

Exercise 1. Let A be a commutative ring. An ideal $\mathfrak{q}$ in A is called *primary* if $ab \in \mathfrak{q}$ implies $a \in \mathfrak{q}$ or $b \in \sqrt{\mathfrak{q}}$.
(a) Show that $\sqrt{\mathfrak{q}}$ is prime if $\mathfrak{q}$ is primary. What about the converse statement?
(b) Show that $\mathfrak{q}$ is primary if $\mathfrak{q}$ is prime. What about the converse statement?
(c) Show that $\mathfrak{q}$ is prime if $\mathfrak{q}$ is maximal. What about the converse statement?

Exercise 2. Use Zorn's Lemma to show the following statements.
(a) Every ideal in a commutative ring is contained in a maximal ideal.
(b) Every cyclic module has a simple quotient.

Exercise 3. Let R be a ring and $Z = Z(R)$ be its center. A derivation is a map $\delta\colon R \to R$ such that $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$.
(a) Show that for $a \in R$ the commutator $x \mapsto ax - xa$ is a derivation.
(b) Show that $\delta(Z) \subset Z$ for any derivation δ.
(c) Show that $\delta(e) = 0$ for $e \in R$ an *idempotent*, that is, $e^2 = e$.

Exercise 4. Let A be a commutative ring. Describe $A[X]^*$. (Do only what was not done in class.)

Exercise 5. Under the hypotheses of the Chinese Remainder Theorem, show that $\prod_{i=1}^n \mathfrak{a}_i = \bigcap_{i=1}^n \mathfrak{a}_i$. Give a counter-example for this equality in general.

Exercise 6. Let A be a commutative ring, and let G be a group.
(a) If $G = \{g_1, \ldots, g_n\}$, show that $g_1 + \cdots + g_n \in Z(A[G])$.
(b) Show that $Z(A[G]) = \{\sum_{g\in G} a_g g \mid \forall h \in G\colon a_g = a_{g^h}\}$.