HOMEWORK 7

Define your terminology and explain notation. If you require a standard result, such as one of the Sylow theorems, then state it (or cite a result from your textbook) before you use it; otherwise give clear and complete proofs of your claims. The problems are of equal value, 6 points each. Partial solutions will be considered on their merits.

Definition 1.

- A covariant functor $F\colon \mathscr{A} \to \mathscr{B}$ is called *faithful* (resp. *full*) if, for all $X, Y \in \mathscr{A}$, the map $F\colon \operatorname{Hom}_{\mathscr{A}}(X,Y) \to \operatorname{Hom}_{\mathscr{B}}(F(X), F(Y))$ is injective (resp. surjective).
- A contravariant functor $G\colon \mathscr{A} \to \mathscr{B}$ is called *faithful* (resp. *full*) if, for all $X, Y \in \mathscr{A}$, the map $G\colon \operatorname{Hom}_{\mathscr{A}}(X,Y) \to \operatorname{Hom}_{\mathscr{B}}(G(Y), G(X))$ is injective (resp. surjective).

Exercise 1.
(a) Give an example of a faithful full functor.
(b) Give an example of a non-faithful full functor.
(c) Give an example of a faithful non-full functor.
(d) Give an example of a non-faithful non-full functor.

Exercise 2. Show that the group $\langle \alpha, \beta \mid \alpha^2 = \beta^2 = 1, \alpha\beta\alpha = \beta\alpha\beta \rangle$ is finite and identify it with a familiar group.

Exercise 3. Find a presentation for the group of integer matrices of the form $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$.

Exercise 4.
(a) Show that the ring $\mathbb{Z}[i]$ is principal.
(b) What are the units?

Exercise 5. Consider the ring of formal power series $A = K[[X]]$ with coefficients in a field K.
(a) Describe the group of units A^*.
(b) Show that A is a domain.
(c) Describe all ideals in A. Which are prime? Which are maximal?

Exercise 6. Let $\phi\colon A \to B$ be a homomorphism of commutative rings.
(a) If $\mathfrak{q}$ is a prime ideal in B then $\phi^{-1}(\mathfrak{q})$ is a prime ideal in A.
(b) Show by example that the analogue of (a) for maximal ideals is false.
(c) If $\mathfrak{m}$ is a maximal ideal in B and ϕ is surjective then $\phi^{-1}(\mathfrak{m})$ is a maximal ideal in A.

Exercise 7. For any set S, denote by $M_n(S)$ of $n \times n$-matrices with entries in S. Let A be a ring. Recall that $M_n(A)$ is a (possibly non-commutative) ring and that "ideal" means "2-sided ideal" in this case.

(a) Show that, for any ideal $\mathfrak{a}$ of A, $M_n(\mathfrak{a})$ is an ideal in $M_n(A)$.
(b) Show that any ideal of $M_n(A)$ is of the form as in (a).
(c) Show that $M_n(A)/M_n(\mathfrak{a}) \cong M_n(A/\mathfrak{a})$.

Exercise 8. Let $\mathfrak{a}$ be an ideal in a commutative ring A and define the *radical* of $\mathfrak{a}$ by

$$\sqrt{\mathfrak{a}} = \{x \in A \mid x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}\}.$$

(a) Show that $\mathfrak{a} \subseteq \mathfrak{b}$ implies $\sqrt{\mathfrak{a}} \subseteq \sqrt{\mathfrak{b}}$.
(b) Show that $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$ and $\sqrt{\mathfrak{a}^n} = \sqrt{\mathfrak{a}}$ for all $n \in \mathbb{N}$.
(c) Show that $\sqrt{\mathfrak{p}} = \mathfrak{p}$ if $\mathfrak{p}$ is a prime ideal of A.