HOMEWORK 5

Define your terminology and explain notation. If you require a standard result, such as one of the Sylow theorems, then state it (or cite a result from your textbook) before you use it; otherwise give clear and complete proofs of your claims. The problems are of equal value, 6 points each. Partial solutions will be considered on their merits.

Exercise 1.

(a) Show that A_{tor} is finitely generated if A is finitely generated.

(b) Show that $A_{\text{tor}} \trianglelefteq A$ and that A/A_{tor} is torsion free.

Exercise 2.

(a) Show that $\mathrm{Hom}(\bigoplus_{i\in\mathbb{N}} \mathbb{Z}, \mathbb{Z}) \cong \prod_{i\in\mathbb{N}} \mathbb{Z}$.

(b) What is $\mathrm{Hom}(\mathbb{Z}, \bigoplus_{i\in\mathbb{N}} \mathbb{Z})$?

Exercise 3. Show that a finite abelian group is not cyclic exactly if it contains a subgroup isomorphic to $Z_p \times Z_p$.

Exercise 4. For finitely generated abelian groups H, K, G show that $G \oplus H \cong G \oplus K$ implies $H \cong K$. Give a counter-example if G is not finitely generated.

Definition 1. Let G be a group.

- A tower of subgroups $1 = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_n = G$ is called a *central series* for G if $[G, G_{i+1}] \leq G_i$.
- Call G *nilpotent* if it admits a central series.
- Define the *upper central series* $Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \cdots \leq G$ of G by $Z_0(G) := 1$ and $Z_{k+1}/Z_k(G) := Z(G/Z_k(G))$ for all $k \geq 0$. (Note that $Z_1(G) = Z(G)$.)
- Define the *lower central series* $G^0 \geq G^1 \geq G^2 \geq \cdots \geq 1$ of G by $G^0 := G$ and $G^{k+1} := [G, G^k]$ for all $k \geq 0$.

Exercise 5.

(a) Show that G is nilpotent if and only if $Z_k(G) = G$ for large k.

(b) Show that G is nilpotent if and only if $G^k = 1$ for large k.

(c) Show that G is solvable if G is nilpotent.

Exercise 6.

(a) Show that D_n is solvable for all n.

(b) Show that D_n is nilpotent if and only if $n = 2^k$ for some k.