HOMEWORK 12

Define your terminology and explain notation. If you require a standard result, such as one of the Sylow theorems, then state it (or cite a result from your textbook) before you use it; otherwise give clear and complete proofs of your claims. The problems are of equal value, 6 points each. Partial solutions will be considered on their merits.

Exercise 1. Prove the intgeral root test.

Exercise 2. Let K be a field of characteristic zero. Show that the multiplicity of the root $a \in K$ in $f \in K[X]$ equals $\min\{k \geq 0 \mid f^{(k)}(a) \neq 0\}$ where $f^{(k)}$ is the kth (formal) derivative of f.

Exercise 3. Show that for $f(X) \in K[X] \leq K[X, Y]$ we have $f(X+Y) = \sum_{k=0}^{\infty} \frac{f^{(k)}(X)}{k!} Y^k$. Note that the sum is finite.

Exercise 4.

(a) Show that the polynomials $X^4 + 1$ and $X^6 + X^3 + 1$ are irreducible over $\mathbb{Q}$.

(b) Show that any degree-3 polynomial over a field is either irreducible, or has a root in the field. Is $X^3 - 5X^2 + 1$ irreducible over $\mathbb{Q}$?

(c) Show that $X^2 + Y^2 - 1$ is irreducible over $\mathbb{Q}$. Is it irreducible over $\mathbb{C}$?

Exercise 5. Let $f = \sum_{i=0}^{n} a_i X^i \in \mathbb{Z}[X]$, $a_n \neq 0$. Suppose that, for some k with $0 < k < n$, $p \not| a_n$, $p \not| a_k$, $p^2 \not| a_0$, but $p | a_i$ for $i < k$. Then show that f has an irreducible factor of degree at least k.

Exercise 6. Let A be an integral domain. Show that any automorphism of $A[X]$ extending id_A is of the form $X \mapsto aX + b$ with $a \in A^*$.

Exercise 7. Let K be a field and $K(X) = Q(K[X])$. Show that any automorphism of $K(X)$ extending id_K is of the form $X \mapsto \frac{aX+b}{cX+d}$ with $a, b, c, d \in K$ such that $ad - bc \in K^*$.