# HOMEWORK 11

Define your terminology and explain notation. If you require a standard result, such as one of the Sylow theorems, then state it (or cite a result from your textbook) before you use it; otherwise give clear and complete proofs of your claims. The problems are of equal value, 6 points each. Partial solutions will be considered on their merits.

**Exercise 1** (Snake Lemma)**.** Prove exactness in the Snake Lemma at the two middle positions.

**Exercise 2** (Graded Modules)**.** Let $A$ be a $K$-algebra with a *compatible filtration* by $K$-subspaces $A_i$, that is, $0 = A_{-1} \leq A_0 \leq A_1 \leq \cdots \leq A$ such that $A_i \cdot A_j \subseteq A_{i+j}$. Recall that $\operatorname{gr} A = \bigoplus_{i=0}^{\infty} A_i/A_{i+1}$ is a $K$-algebra in this case.

(a) Show that $\operatorname{gr} M = \bigoplus_{i=0}^{\infty} M_i/M_{i+1}$ is a $\operatorname{gr} A$-module. Let $M$ be an $A$-module with a *compatible filtration* by $K$-subspaces $M_i$, that is, $0 = M_{-1} \leq M_0 \leq M_1 \leq \cdots \leq M$ such that $A_i \cdot M_j \subseteq M_{i+j}$.

(b) Assume that $M$ and $N$ are two modules as in (a), and $\phi\colon M \to N$ an $A$-linear map such that $\phi(M_i) \leq N_i$. Define $\operatorname{gr}\phi\colon \operatorname{gr} M \to \operatorname{gr} N$.

(c) Show that gr is a covariant functor, and that it is not exact in general. It is exact if the maps are *strict*. Can you guess what the definition of strictness is?

**Exercise 3** (Newton Filtration and Kouchnirenko Resolution)**.** Let $A = K[X, Y] = K[\mathbb{N}^2]$. Consider $\mathbb{N}^2 \leq \mathbb{Q}^2$ as a sub-semigroup. Consider the polytope (unit square) $\Delta = [0, 1] \times [0, 1] \leq \mathbb{Q}^2$.

(a) The *Newton filtration* on $A$ with respect to $\Delta$ is defined by

$$N_k = \left\langle X^\alpha Y^\beta \mid (\alpha, \beta) \in \mathbb{N}^2 \cap k \cdot \Delta \right\rangle_K.$$

Show that $N_k$ is a compatible filtration. Write $\operatorname{gr} A$ for $\operatorname{gr}^N A$ in the following.

(b) Consider the *facets* $\sigma_1 = \{1\} \times [0, 1]$ and $\sigma_2 = [0, 1] \times \{1\}$ and the *face* $\sigma_0 = \sigma_1 \cap \sigma_2 = \{(1, 1)\}$ of $\Delta$. These define sub-semigroups $S_i = \mathbb{N}^2 \cap (\mathbb{Q}_{\geq 0} \cdot \sigma_i)$ of $\mathbb{N}^2$, and sub-semigroup rings $A_i = K[S_i]$ of $A$. Show that the $A_i = \operatorname{gr} A_i$ are $\operatorname{gr} A$-modules, where $\operatorname{gr} A_i$ is defined using the induced filtration $N_k \cap A_i$ on $A_i$. Note that the $A_i$ are not $A$-modules.

(c) Show that there is an exact sequence of $\operatorname{gr} A$-modules

$$0 \to A_0 \to A_1 \oplus A_2 \to \operatorname{gr} A \to 0.$$

This exercise illustrates what I said in class: The Newton-graded polynomial ring $\operatorname{gr} A$ consists of semigroup rings $A_i$ "patched together". Such rings show up in the context of so called *toric varieties*. A.G. Kouchnirenko used a generalization of (c) to compute Milnor numbers of certain singularities in terms of volumes of polytopes.

**Exercise 4** (Weyl Algebra)**.** The (non-commutative) *Weyl algebra* $A_n$ is defined as the associative $K$-algebra generated by variables $x_1, \dots, x_n$ and $\partial_1, \dots, \partial_n$ subject to the relations $[x_i, x_j] = 0 = [\partial_i, \partial_j]$ and $[\partial_i, x_j] = \delta_{i,j}$ where $[a, b] = ab - ba$ is the commutator.

To understand the philosophy, let $P = K[x_1, \dots, x_n]$, and consider multiplication by $x_i$ and the partial derivation $\partial_j = \frac{\partial}{\partial x_j}$ as elements of $\mathrm{End}_K(P)$. This turns $A_n$ into a sub-$K$-algebra of $\mathrm{End}_K(P)$: For instance,

$$\partial_j(x_j(f)) - x_j(\partial_j(f)) = f,$$

for $f \in P$, by the product rule, whichexplains the commutator relation $[\partial_j, x_j] = 1$.

(a) Note that a general element of $A_n$ is a finite $K$-linear combination of finite products involving $x_i$ and $\partial_j$. For $P \in A_n$, the *order* $\mathrm{ord}\, P$ of $P$ is defined as the maximum number of $\partial_j$ in such a product. Show that $\mathrm{ord}\, P$ is well defined.

(b) Let $F_k$ be the increasing filtration by order $k$ on $A_n$. Show that $\mathrm{gr}\, A_n$ is a (commutative) polynomial ring $K[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ where $\xi_j$ is the class of $\partial_j$ in $F_1/F_0$.

(c) The subring $P \leq A_n$ can be identified with $A_n$ modulo the left ideal $\langle \partial_1, \dots, \partial_n \rangle$ and is therefore a left $A_n$-module. Consider a left ideal $\mathfrak{a} = \langle P_1, \dots, P_n \rangle \in A_n$. Show that $\mathrm{Hom}_{A_n}(A_n/\mathfrak{a}, P)$ can be identified with the ($K$-vector space of) solutions $f \in P$ of the system of linear partial differential equations $P_i(f) = 0$, $i = 1, \dots, n$.

What you see in this exercise is the starting point of $D$-module theory, the algebraic theory of linear partial differential equations.