HOMEWORK 10

Define your terminology and explain notation. If you require a standard result, such as one of the Sylow theorems, then state it (or cite a result from your textbook) before you use it; otherwise give clear and complete proofs of your claims. The problems are of equal value, 6 points each. Partial solutions will be considered on their merits.

Exercise 1 (Rank of free modules)**.** Let A be a ring.

(a) Show uniqueness of n in $M \cong A^n$, that is, the rank $\operatorname{rk} M = n$ of M is well-defined. (Hint: Use that the dimension of a vector space is well defined.)

(b) Show that a finitely generated free A-module M has a finite basis, that is, it is isomorphic to A^n for some n. (Hint: You could use that M is projective and apply another vector space argument.)

Exercise 2 (Zero-divisors and associated primes)**.** Let R be a commutative ring. Assume that R is Noetherian, that is, any increasing chain of ideals is stationary. A prime ideal $\mathfrak{p} \leq R$ is an *associated prime* of R if there is a monomorphism $R/\mathfrak{p} \hookrightarrow R$ of R-modules. For $x \in R$, the *annihilator* $\operatorname{Ann}(x)$ of x is the kernel of multiplication by x considered as R-linear map $R \to R$.

(a) Show that any ideal that is maximal among the $\operatorname{Ann}(x)$ for some $0 \neq x \in A$ is a prime ideal.

(b) Conclude that the set of zero-divizors equals the union of associated primes.

If you believe that minimal primes are associated, this generalizes what I explained in class: non-zero elements of minimal primes are zero-divisors.

Exercise 3 (The determinant trick)**.** Let M be a finitely generated R-module and let $\mathfrak{a}$ be an ideal in R.

(a) Show that any $\phi \in \operatorname{End}_R(M)$ with $\phi(M) \leq \mathfrak{a}M$ is *integral* over $\mathfrak{a}$, that is, $\phi^n + a_1\phi^{n-1} + \cdots + a_{n-1}\phi + a_n = 0$ where $a_i \in \mathfrak{a}^i$ for $i = 1, \ldots, n$. Hint: Represent ϕ by a matrix A and consider $\det(\phi \cdot I - A) \in \operatorname{End}_R(M)$ where I is a unit matrix.

(b) Show that $\mathfrak{a}M = M$ iff $(1 - a)M = 0$ for some $a \in \mathfrak{a}$. Hint: Apply (a) to $\phi = \mathrm{id}_M$.

Exercise 4 (Vasconcelos-Strooker theorem)**.** Let M be a finitely generated module over a commutative ring R. Show that $\phi \in \operatorname{End}_R(M)$ is injective if it is surjective. Hint: Consider M as $R[X]$-module and apply determinant trick to $\mathfrak{a} = \langle X \rangle$. Note also that by exchanging “injective” and “surjective” you break the analogy with the linear algebra case. Can you find a counter-example?

Exercise 5 (Jacobson radical and Nakayama’s lemma)**.** The *Jacobson radical* $\operatorname{Rad}(R)$ of a commutative ring R is the intersection of all maximal ideals.

(a) Show that $x \in \mathrm{Rad}(R)$ iff $1 + xy \in R^*$ for all $y \in R$.

(b) Let $\mathfrak{a} \leq \mathrm{Rad}(R)$ and M a finitely generated R-module. Then $\mathfrak{a}M = M$ implies $M = 0$. Hint: Use determinant trick.

Exercise 6 (Ideals generated by idempotents)**.** Let $\mathfrak{a}$ be an ideal in a commutative ring R. Assume that $\mathfrak{a}$ is finitely generated, which is automatic if R is Noetherian. Show that $\mathfrak{a} = \mathfrak{a}^2$ iff $\mathfrak{a} = \langle e \rangle$ for some idempotent $e \in \mathfrak{a}$. Hint: Apply the determinant trick to $M = \mathfrak{a}$.