HOMEWORK 1

Define your terminology and explain notation. If you require a standard result, such as one of the Sylow theorems, then state it (or cite a result from your textbook) before you use it; otherwise give clear and complete proofs of your claims. The problems are of equal value, 5 points each. Partial solutions will be considered on their merits.

Exercise 1.

(a) Prove uniqueness of the unit element and the inverse of each element of a group.

(b) Show that a monoid such that each element admits a left inverse is a group.

(c) Show that group homomorphisms commute with taking the inverse.

Exercise 2. Let G be a group. Show that the following are equivalent:

(a) G is abelian.

(b) $x \mapsto x^2$ defines a homomorphism $G \to G$.

(c) $x \mapsto x^{-1}$ defines a homomorphism $G \to G$.

(d) $x \mapsto x^n$ defines a homomorphism $G \to G$ all $n \in \mathbb{Z}$.

(e) $x \mapsto x^n$ defines a homomorphism $G \to G$ for three consecutive $n \in \mathbb{Z}$.

Exercise 3.

(a) Let G, H be groups and $S \subseteq H$ be a subset. Then any homomorphism $\phi\colon \langle S \rangle \to G$ is uniquely determined by its restriction $\phi|_S$ to S (which is just a map of sets $S \to G$).

(b) Show that the kernel and image of a homomorphism are (sub)groups. Relate kernel and image to injectivity and surjectivity.

Exercise 4.

(a) Show that the symmetric group S_n is generated by transpositions.

(b) Show that $|S_n| = n!$.

(c) Show that for p prime, S_p is generated by any pair of a 2- and a p-cycle.

Exercise 5. Let H and K be subgroups of a finite group G. Show that:

(a) $|HK| = \frac{|H||K|}{|H \cap K|}$.

(b) HK is a subgroup if and only if $HK = KH$.

Exercise 6. Let G be a group.

(a) If G/C is cyclic for a subgroup $C \leq Z_G$, then show that G is abelian.

(b) If $\mathrm{Aut}(G)$ is cyclic, then show that G is abelian.

Exercise 7. Let K and H be finite-index subgroups of a group G.

(a) Show that $K \cap H$ is a finite-index subgroup of G.

(b) Show that there exists a finite-index normal subgroup N of G which is contained in H.

Exercise 8. Let $1 \longrightarrow K \xrightarrow{\psi} G \xrightarrow{\varphi} H \longrightarrow 1$ be a short exact sequence of groups. Assume that it *splits*, which means that there is a map $G \xleftarrow{\alpha} H$ such that $\varphi \circ \alpha = \mathrm{id}_H$. Show that $G = \mathrm{Im}\,\psi \rtimes \mathrm{Im}\,\alpha \cong K \rtimes H$.