First Name:______________________ **Last Name:**______________________

OSU Number:____________________ **Signature:**___________________________

OKLAHOMA STATE UNIVERSITY
Department of Mathematics

MATH 2144 (Calculus I)
Instructor: Dr. Mathias Schulze

MIDTERM 1
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Duration: 50 minutes

No aids allowed.

This examination paper consists of **6** pages and **5** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer **4 (incl. #1)** questions.

To obtain credit, you must give arguments to support your answers.

For graders' use:

	Score
1 (10)	
2 (10)	
3 (10)	
4 (10)	
5 (10)	
Total (50)	

1. [10] True or False? Write a "T" (for true) or an "F" (for false) for each statement.

   (a) $\lim_{x\to 4}\left(\frac{2x}{x-4}-\frac{8}{x-4}\right)=\lim_{x\to 4}\frac{2x}{x-4}-\lim_{x\to 4}\frac{8}{x-4}$

   (b) If p is a polynomial, then $\lim_{x\to 1} p(x) = p(1)$.

   (c) If $\lim_{x\to a}[f(x)g(x)]$ exists, then it must be equal to $f(a)g(a)$.

   (d) If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = -\infty$, then $\lim_{x\to a}[f(x)+g(x)] = 0$.

   (e) If $x = 1$ is a vertical asymptote of $y = f(x)$ then f is not defined at 1.

   (f) If f is continuous at a, then f is differentiable at a.

   (g) If $f(x) > 1$ for all $x > 0$ and $\lim_{x\to 0^+} f(x)$ exists, then $\lim_{x\to 0^+} f(x) > 1$.

   (h) If $f'(r)$ exists, then $\lim_{x\to r} f(x) = f(r)$.

   (i) The equation $x^{10} - 10x^2 + 5 = 0$ has a solution in the interval $(0, 2)$.

   (j) A rational function can have two different horizontal asymptotes.

   Solution: (a) F (limit law does not apply to infinite limits)

   (b) T (polynomials are continuous)

   (c) F (example: $f(x) = x$, $g(x) = x^{-1}$, $a = 0$)

   (d) F (example: $f(x) = x^{-2}$, $g(x) = -x^{-4}$, $a = 0$)

   (e) F (limits in the definition of the vertical asymptote "ignore" $f(1)$)

   (f) F (example: $f(x) = |x|$)

   (g) F (example: $f(x) = x + 1$)

   (h) T (differentiable implies continous)

   (i) T (apply Intermediate Value Theorem to $[0, 1]$)

   (j) F (rational functions have at most one, algebraic functions can have two)

2. [10] Compute the limits.

   (a) $\lim_{x\to\pi} \sin(x + \sin(x + \sin(x + \sin(x + \sin x))))$
   (b) $\lim_{x\to\frac{\pi}{8}} \arctan\left(\frac{64x^2-\pi^2}{64x-8\pi}\right)$
   (c) $\lim_{x\to-\infty} \frac{\sqrt{4x^6-x}}{x^3+9}$
   (d) $\lim_{t\to\infty} \frac{t^2-t}{2t^2+t+7}$
   (e) $\lim_{x\to 0} \left(x^4 \cos \frac{2}{x}\right)$ (Hint: use the Squeeze Theorem)

Solution:

   (a) By continuity, $\lim_{x\to\pi} \sin(x+\sin(x+\sin(x+\sin(x+\sin x)))) = \sin(\pi+\sin(\pi+\sin(\pi + \sin(\pi + \sin \pi)))) = 0$.
   (b) By Theorem 8 in Section 2.5, $\lim_{x\to\frac{\pi}{8}} \arctan\left(\frac{64x^2-\pi^2}{64x-8\pi}\right) = \arctan\left(\lim_{x\to\frac{\pi}{8}} \frac{64x^2-\pi^2}{64x-8\pi}\right)$. But $\lim_{x\to\frac{\pi}{8}} \frac{64x^2-\pi^2}{64x-8\pi} = \lim_{x\to\frac{\pi}{8}}(x+\frac{\pi}{8}) = \frac{\pi}{4}$ and hence the result is $\arctan\frac{\pi}{4} = 1$.
   (c) Note that for $x < 0$, we have $x^{-3} = -\sqrt{x^{-6}}$. Using this, we compute $\lim_{x\to-\infty} \frac{\sqrt{4x^6-x}}{x^3+9} = \lim_{x\to-\infty} -\frac{\sqrt{4x^6-x}\sqrt{x^{-6}}}{(x^3+9)x^{-3}} = \lim_{x\to-\infty} -\frac{\sqrt{4-\frac{1}{x^5}}}{1+\frac{9}{x^3}} = -2$.
   (d) $\lim_{t\to\infty} \frac{t^2-t}{2t^2+t+7} = \lim_{t\to\infty} \frac{t^2}{2t^2} = \frac{1}{2}$
   (e) We have $-x^4 \le x^4 \cos\frac{2}{x} \le x^4$ and $\lim_{x\to 0} x^4 = 0$. So, by the Squeeze Theorem, it follows that also $\lim_{x\to 0}\left(x^4 \cos\frac{2}{x}\right) = 0$.

3. [10]

(a) For $f(x) = \frac{2x^2-18}{x^2+2x-3}$, find all asymptotes and the limits that describe the asymptotic behavior of the function.

(b) Find the horizontal asymptotes of the function $f(x) = \frac{\sqrt{x^6-1}}{x^3+7x^2+4x-8}$.

Solution: (a) Dropping the terms with not highest exponents in the numerator and denominator of f (as explained in the lecture) yields $y = \frac{2x^2}{x^2} = 2$ as horizontal asymptote for both $x \to \infty$ and $x \to -\infty$. So we have $\lim_{x\to\pm\infty} f(x) = 2$.
To find the vertical asymptotes, we factorize and cancel factors if possibe:

$$f(x) = \frac{2x^2 - 18}{x^2 + 2x - 3} = 2\frac{(x+3)(x-3)}{(x+3)(x-1)} = 2\frac{x-3}{x-1}$$

So, $x = 1$ is the only vertical asymptote. As $x - 3 < 0$ for x close to 1, we have $\lim_{x\to 1^-} = \infty$ and $\lim_{x\to 1^+} = -\infty$.

(b) Dropping the terms with not highest exponents under the root and in the denominator of f (as above) yields $\frac{\sqrt{x^6}}{x^3} = \frac{|x|^3}{x^3} = |x|/x$ which has the same asymptotic behavior as $f(x)$ for large $|x|$. So $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} |x|/x = 1$ and similarly $\lim_{x\to-\infty} f(x) = -1$. In other words, $y = 1$ and $y = -1$ are two (different) horizontal asymptotes.

4. [10]

(a) Find all values for a and b such that the function

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if } x < 1 \\ (x-a)^2 & \text{if } 1 \leq x < 2 \\ 2ax - b & \text{if } 2 \leq x \end{cases}$$

becomes continuous.

(b) Is f differentiable for some choice of a and b?

Solution:

(a) First, note that $f(x) = x + 3$ for $x < 1$. Continuity is clear at $x \neq 1, 2$, as polynomials are continuous. The following two conditions are equivalent to continuity at 1 and 2 respectively:

$$4 = \lim_{x\to 1^-} f(x) = f(1) = (1-a)^2,$$
$$(2-a)^2 = \lim_{x\to 2^-} f(x) = f(2) = 4a - b.$$

The first equality gives $a = 1 \mp 2$, so $a = -1$ or $a = 3$. Then the second equality reads $5 \pm 4 = (1 \pm 2)^2 = 4 \mp 8 - b$ which gives $b = -1 \mp 12$. So either $a = -1$ and $b = -13$ or $a = 3$ and $b = 11$.

(b) Note that if f is continuous, then $f(1) = 4$ by the first part. For $f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$ to exist, the corresponding left- and right-sided limits

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{1+h+3-4}{h} = 1,$$
$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{(1+h-a)^2-4}{h}$$

must be equal. But, for $a = 1 \mp 2$, the right-sided limit equals

$$\lim_{h\to 0^+} \frac{(h\pm 2)^2-4}{h} = \lim_{h\to 0^+} \frac{h^2 \pm 4h}{h} = \pm 4$$

which is not equal to the left-sided limit. So the answer is “no”.

5. [10]

   (a) Compute the derivative of $f(x) = \frac{1-x}{1+x}$ (using the limit definition).

   (b) Find the equation of the tangent line to the graph of f at the point $(1, 0)$.

Solution:

(a)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1-x-h}{1+x+h} - \frac{1-x}{1+x}}{h} \\
&= \lim_{h\to 0} \frac{(1-x-h)(1+x) - (1-x)(1+x+h)}{h(1+x+h)(1+x)} \\
&= \lim_{h\to 0} \frac{1-x-h+x-x^2-hx-1-x-h+x+x^2+hx}{h(1+x+h)(1+x)} \\
&= \lim_{h\to 0} \frac{-2h}{h(1+x+h)(1+x)} = -\frac{2}{(1+x)^2}
\end{aligned}$$

(b) The slope is $m = f'(0) = -\frac{1}{2}$, so the equation reads $y = m(x-1) = \frac{1}{2} - \frac{1}{2}x$.

End of examination
Total pages: 6
Total marks: 50