First Name:________________________ **Last Name:**________________________

OSU Number:______________________ **Signature:**______________________________

OKLAHOMA STATE UNIVERSITY
Department of Mathematics

MATH 2163 (Calculus III)
Instructor: Dr. Mathias Schulze

MIDTERM 3
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Duration: 50 minutes

No aids allowed.

This examination paper consists of **6** pages and **5** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer question 1 and pick 3 from the following questions.

To obtain credit, you must give arguments to support your answers.

For graders' use:

	Score
1 (24)	
2 (8)	
3 (8)	
4 (8)	
5 (8)	
Total (56)	

1. [24]

   (a) Write down a formula for the average value f_{av} of a function $f(x, y, z)$ on a region D.

   (b) Write down a formula for the mass m of a solid S with density $\rho(x, y, z)$.

   (c) Write down a formula for the x-coordinate $\bar{x}$ of the center of mass of a solid S with density $\rho(x, y, z)$.

   (d) Write down a formula for the moment of inertia I_x about the x-axis of a lamina with density $\rho(x, y)$ that occupies the region D.

   (e) Write down the formulæ to transform cylindrical coordinates r, θ, z into cartesian coordinates x, y, z.

   (f) What is $dV = dx\, dy\, dz$ in terms of cylindrical coordinates?

   (g) Write down the formulæ to transform spherical coordinates ρ, θ, ϕ into cartesian coordinates x, y, z.

   (h) What is $dV = dx\, dy\, dz$ in terms of spherical coordinates?

   Solution: See textbook.

2. [8] Evaluate the triple integral $\iiint_E z\,dV$, where E is the region bounded by $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Solution: This is a simplified version of Problem 15.6.12; its solution has been explained in the review session.

The region E is defined by the inequalities $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x+y+z \leq 1$. So the lower bound for all three variables is 0. The upper bound for z is 1 (attained when $x = 0 = y$). For fixed z, the upper bound for y is $1 - z$ (attained when $x = 0$). For fixed y and z, the upper bound for x is $1 - y - z$. Therefore

$$\begin{aligned}
\iiint_E z\,dV &= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} z\,dx\,dy\,dz \\
&= \int_0^1 z \int_0^{1-z} 1 - y - z\,dy\,dz \\
&= \int_0^1 z\left[y - y^2/2 - yz\right]_0^{1-z} dz \\
&= \int_0^1 z(1 - z - 1/2 + z - z^2/2 - z + z^2)\,dz \\
&= \int_0^1 z/2 - z^2 + z^3/2\,dz \\
&= [z^2/4 - z^3/3 + z^4/8]_0^1 \\
&= 1/24.
\end{aligned}$$

3. [8] Use cylindrical coordinates to evaluate the integral $\iiint_E e^z\, dV$, where E is inclosed by the paraboloid $z = 1 + x^2 + y^2$, the cylinder $x^2 + y^2 = 5$, and the x-y-plane.

Solution: This is Problem 15.7.19; its solution has been explained in the review session.

In cylindrical coordinates, the solid is defined by $0 \leq z \leq 1 + r^2$ (since $z = 0$ defines the x-y-plane), and $r = \sqrt{5}$ (since r is always positive). Note that there are no conditions on θ, so $0 \leq \theta \leq 2\pi$. Thus,

$$\begin{aligned}\iiint_E e^z\, dV &= \int_0^{2\pi} \int_0^{\sqrt{5}} \int_0^{1+r^2} e^z r\, dz\, dr\, d\theta \\ &= \pi \int_0^{\sqrt{5}} 2re^{1+r^2} - 2r\, dr \\ &= \pi \left[e^{1+r^2} - r^2 \right]_0^{\sqrt{5}} \\ &= \pi(e^6 - e - 5).\end{aligned}$$

4. [8] Use spherical coordinates to evaluate the integral $\iiint_H x^2 + y^2\, dV$ where H is the solid hemisphere $x^2 + y^2 + z^2 \leq 9$, $z \geq 0$.

Solution: This is a simplified version of Problem 15.8.22; its solution has been explained in the review session.

The hemisphere H is defined by the inequalities $\rho \leq 3$ and $\phi \leq \pi/2$, with no conditions on θ. Note also that $x^2 + y^2 = \rho^2 \sin^2 \phi$ in spherical coordinates. Therefore

$$\begin{aligned}\iiint_H x^2 + y^2\, dV &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^3 \rho^2 \sin^2 \phi \cdot \rho^2 \sin \phi\, d\rho\, d\phi\, d\theta \\ &= 2\pi \int_0^3 \rho^4\, d\rho \int_0^{\pi/2} \sin^3 \phi\, d\phi \\ &= 2\pi \frac{3^5}{5} \int_0^{\pi/2} (1 - \cos^2 \phi) \sin \phi\, d\phi \\ &= 2\pi \frac{3^5}{5} \left[-\cos \phi + \frac{1}{3} \cos^3 \phi \right]_0^{\pi/2} \pi \frac{3^5}{5} (1 - 1/3) = \frac{4\pi 3^4}{5}.\end{aligned}$$

5. [8] Use the transformation $x = 2u+v$, $y = u+2v$, to evaluate the integral $\iint_R x-3y\,dA$, where R is the triangular region with vertices $(0,0)$, $(2,1)$, $(1,2)$.

Solution: This is Problem 15.9.11; its solution has been explained in the review session.
Solving for u and v, we compute the inverse transformation: $u = \frac{2x-y}{3}$, $v = \frac{2y-x}{3}$. As the transformation is linear, the image of R is the triangle S with vertices $(0,0)$, $(1,0)$, $(0,1)$, obtained by applying the inverse transformation to the vertices of R. Transforming the function gives $x-3y = 2u+v-3u-6v = -u-5v$. The Jacobian of the transformation is the determinant

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 3.$$

Finally, we can compute

$$\begin{aligned}
\iint_R x-3y\,dA &= -3\iint_S u+5v\,du\,dv \\
&= -3\int_0^1\int_0^{1-v} u+5v\,du\,dv \\
&= -3\int_0^1 \left[\frac{1}{2}u^2+5vu\right]_0^{1-v} dv \\
&= -3\int_0^1 \frac{1}{2}+4v-\frac{9}{2}v^2\,dv \\
&= -3\left(\frac{1}{2}+2-\frac{3}{2}\right) = -3.
\end{aligned}$$

End of examination
Total pages: 6
Total marks: 56