First Name:____________________ **Last Name:**____________________

OSU Number:____________________ **Signature:**____________________

OKLAHOMA STATE UNIVERSITY
Department of Mathematics

MATH 2163 (Calculus III)
Instructor: Dr. Mathias Schulze

FINAL EXAM
December 11, 2009

Duration: 90 minutes

No aids allowed.

This examination paper consists of **10** pages and **8** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer questions 1-3 and pick 3 from the following questions.

To obtain credit, you must give arguments to support your answers.

For graders' use:

	Score
1 (9)	
2 (12)	
3 (8)	
4 (6)	
5 (6)	
6 (6)	
7 (6)	
8 (6)	
Total (59)	

1. [9] For the vectors $\mathbf{a} = \langle 1, 1, 1\rangle$ and $\mathbf{b} = \langle 1, 2, 3\rangle$ compute

   (a) the dot product $\mathbf{a} \cdot \mathbf{b}$,

   (b) the cross product $\mathbf{a} \times \mathbf{b}$, and

   (c) the vector projection of $\mathbf{a}$ in direction of $\mathbf{b}$.

   Solution:

   (a) $\mathbf{a} \cdot \mathbf{b} = 1 + 2 + 3 = 6$

   (b) $\mathbf{a} \times \mathbf{b} = \langle 3 - 2, 1 - 3, 2 - 1\rangle = \langle 1, -2, 1\rangle$

   (c) $\mathbf{proj_b}(\mathbf{a}) = (\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{b}/|\mathbf{b}|^2 = 3/7 \cdot \langle 1, 2, 3\rangle$.

2. [12]

(a) Find parametric equations for the line through $(2, 1, 0)$ and perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$.

(b) Find an equation of the plane that passes through the point $(6, 0, -2)$ and contains the line $x = 4 - 2t$, $y = 3 + 5t$, $z = 7 + 4t$.

(c) Find the tangent plane to the surface $z = x^3y^4$ at the point $(1, 1, 1)$.

Solution: (a) A direction vector is given by $(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + \mathbf{k}) = \mathbf{i} - \mathbf{j} + \mathbf{k}$, so the components of

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} = (2 + t)\mathbf{i} + (1 - t)\mathbf{j} + t\mathbf{k}, \quad t \in \mathbb{R},$$

are the parametric equations of the line in question.

(b) The point $(4, 3, 7)$ is in the line, so the vector $\langle 4, 3, 7\rangle - \langle 6, 0, -2\rangle = \langle -2, 3, 0\rangle$ is in the plane. Then a normal vector is obtained as

$$\langle -2, 3, 9\rangle \times \langle -2, 5, 4\rangle = \langle -33, -10, -4\rangle$$

and the equation of the plane reads $33x + 10y + 4z = 6 \cdot 33 - 8 = 190$.

(c) For $f(x, y) = x^3y^4$, $\nabla f(1, 1) = \langle 3, 4\rangle$ and hence $z = 1 + 3(x - 1) + 4(y - 1) = 3x + 4y - 6$ defines the tangent plane.

3. [8]

   (a) Use the chain rule to find $\partial z/\partial s$ where $z = e^{x+2y}$, $x = s/t$, $y = t/s$.

   (b) Use implicit differentiation to compute dy/dx where $\sqrt{xy} = 1 + x^2 y$.

Solution:

(a) $\partial z/\partial s = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = z/t - 2zt/s^2$

(b) Applying d/dx to the equality gives $\frac{y+xy'}{2\sqrt{xy}} = 2xy + x^2 y'$, and, by solving for y', we find $y' = \frac{2x - \frac{1}{2\sqrt{xy}}}{x(\frac{1}{2\sqrt{xy}} - x)} y = \frac{4x\sqrt{xy} - 2}{x(1 - 2x\sqrt{xy})} y$.

4. [6] Consider the function $f(x,y) = xy + \frac{1}{x} + \frac{1}{y}$.

   (a) Find all critical points and determine whether they are local maxima, local minimal, or saddle points.

   (b) Are there any global maxima or minima? Explain why.

Solution:

(a) Setting $\nabla f = \langle y - 1/x^2, x - 1/y^2 \rangle = 0$ gives $x = y = 1$. The Hessian at this point is $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, so it is a local minimum by the 2nd derivative test.

(b) Since $\lim_{x \to 0\pm} f(x,1) = \pm\infty$, there is no global maximum or minimum.

5. [6] Find the maximum and minimum values of the function $f(x,y) = 2x^2+3y^2-4x-5$ subject to the constraint $x^2+y^2 \leq 16$.

Solution: To find critcal points (x,y) with $x^2+y^2 < 16$, set $\nabla f = \langle 4x-4, 6y\rangle = 0$. This gives $x = 1$ and $y = 0$, and $f(1,0) = -7$. To find critcal points with $g = x^2+y^2-16 = 0$, apply Lagrange multipliers: Using $\nabla g = \langle 2x, 2y\rangle$, the system of equations $\nabla f = \lambda\nabla g$, $g = 0$, becomes: $2x-2 = \lambda x$, $3y = \lambda y$, $x^2+y^2 = 16$. Rewrite the first two equations as $(2-\lambda)x = 2$, $(3-\lambda)y = 0$. If $y = 0$ then $x = \pm 4$, and the corresponding values of f are 11 and 43. Otherwise, $\lambda = 3$, $x = -2$, $y = \pm 2\sqrt{3}$, and the value of f is 47. Thus, f has minimum value -7 at $(1,0)$, and maximum value 47 at $(-2, \pm 2\sqrt{3})$.

6. [6] Find the centroid (= center of mass) of a solid homogeneous hemisphere H of radius a.

Solution: We can assume that the density equals 1. Then the mass is $m = V(H) = \frac{2}{3}\pi a^3$. We choose a coordinate system such that the origin is the center of the complete sphere and such that H is symmetric with respect to rotation about the z-axis. Then, by symmetry $\bar{x} = \bar{y} = 0$, where $(\bar{x}, \bar{y}, \bar{z})$ denotes the centroid. To compute $\bar{z}$ first compute

$$
\begin{aligned}
\int_H z\,dV &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^a \rho\cos\phi \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= 2\pi \int_0^{\frac{\pi}{2}} \sin\phi\cos\phi\, d\phi \cdot \int_0^a \rho^3\, d\rho \\
&= \frac{\pi}{2}a^4 \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin(2\phi)\, d\phi \\
&= \frac{\pi}{4}a^4 \left(\frac{1}{2}\cos(\phi)|_{\pi}^{0}\right) \\
&= \frac{\pi}{4}a^4.
\end{aligned}
$$

Thus, $\bar{z} = \frac{1}{m}\int_H z\,dV = \frac{3}{8}\pi a$ and the centroid is $(0, 0, \frac{3}{8}a)$.

7. [6]

(a) Evaluate the line integral $\int_C xy\,ds$ where C is the curve given by $x = t^2$, $y = 2t$, $0 \le t \le 1$. Hint: substitute $t = \tan\phi$. You do not need to simplify the result of the integration.

(b) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ and C is given by the vector function $\mathbf{r}(t) = t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}$, $0 \le t \le \pi$.

Solution: (a)

$$\begin{aligned}
\int_C xy\,ds &= 4\int_0^1 t^3\sqrt{t^2+1}\,dt \\
&= \int_0^{\pi/4} \tan^3\phi\sqrt{\tan^2\phi+1}\sec^2\phi\,d\phi \\
&= \int_0^{\pi/4} \frac{\sin^3\phi}{\cos^6\phi}\,d\phi \\
&= \int_0^{\pi/4} \frac{\sin\phi}{\cos^6\phi} - \frac{\sin\phi}{\cos^4\phi} \\
&= -\left[\frac{1}{5}\cos^{-5}\phi + \frac{1}{3}\cos^{-3}\phi\right]_0^{\pi/4}
\end{aligned}$$

(b)

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{\pi} \langle \cos t, \sin t, t\rangle \cdot \langle 1, \cos t, -\sin t\rangle\,dt \\
&= \int_0^{\pi} \cos t + \frac{1}{2}\sin(2t) - t\sin t\,dt \\
&= \left[\sin t - \frac{1}{4}\cos(2t) + t\cos t + \sin t\right]_0^{\pi} \\
&= -\pi
\end{aligned}$$

8. [6]

(a) Show that the force field $\mathbf{F} = \langle e^{-y}, -xe^{-y} \rangle$ is conservative.

(b) Find a potential Φ for $\mathbf{F}$, that is, a function Φ such that $\mathbf{F} = \nabla\Phi$.

(c) Compute the work W done by the force field $\mathbf{F}$ in moving an object from $P(0,1)$ to $Q(2,0)$. Recall that $W = \int_C \mathbf{F} \cdot d\mathbf{r}$ is the work done by $\mathbf{F}$ in moving an object along the curve C.

Solution:

(a) Writing $\mathbf{F} = \langle p, q \rangle$ we have $p = e^{-y}$, and $q = -xe^{-y}$. Then p and q have continuous first order derivatives, and $p_q = -e^{-y} = q_x$. Since the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is simply connected and open, this impies that $\mathbf{F}$ is conservative.

(b) We need to solve the system of differential equations $\langle e^{-y}, -xe^{-y} \rangle = \mathbf{F} = \nabla\Phi = \langle \Phi_x, \Phi_y \rangle$, or, equivalently,

$$\Phi_x = e^{-y}, \quad \Phi_y = -xe^{-y}.$$

Integrating the first equation with respect to x gives $\Phi = xe^{-y} + g(y)$. Then plugging into the second, shows that $-xe^{-y} = \Phi_y = -xe^{-y} + g'(y)$ and hence that $g(y) = c$ must be constant, which we may assume to be $c = 0$. One easily checks that $\Phi = xe^{-y}$ is indeed a solution.

(c) We apply the fundamental theorem of line integrals using the potential Φ from (b). Pick a curve C from P to Q. Then $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \Phi(Q) - \Phi(P) = 2$.

End of examination
Total pages: 10
Total marks: 59