First Name:____________________ **Last Name:**____________________

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OKLAHOMA STATE UNIVERSITY
Department of Mathematics

MATH 2144 (Calculus I)
Instructor: Dr. Mathias Schulze

MIDTERM 1
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Duration: 50 minutes

No aids allowed.

This examination paper consists of **7** pages and **6** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer **5 of 6** questions.

To obtain credit, you must give arguments to support your answers.

For graders' use:

	Score
1 (10)	
2 (10)	
3 (10)	
4 (10)	
5 (10)	
6 (10)	
Total (60)	

1. [10] True or False? Write a "T" (for true) or an "F" (for false) for each statement.

   (a) $\lim_{x\to 4}\left(\frac{2x}{x-4}-\frac{8}{x-4}\right)=\lim_{x\to 4}\frac{2x}{x-4}-\lim_{x\to 4}\frac{8}{x-4}$

   (b) If p is a polynomial, then $\lim_{x\to 1}p(x)=p(1)$.

   (c) If $\lim_{x\to a}[f(x)g(x)]$ exists, then it must be equal to $f(a)g(a)$.

   (d) If $\lim_{x\to a}f(x)=\infty$ and $\lim_{x\to a}g(x)=-\infty$, then $\lim_{x\to a}[f(x)+g(x)]=0$.

   (e) If $x=1$ is a vertical asymptote of $y=f(x)$ then f is not defined at 1.

   (f) If f is continuous at a, then f is differentiable at a.

   (g) If $f(x)>1$ for all $x>0$ and $\lim_{x\to 0^+}f(x)$ exists, then $\lim_{x\to 0^+}f(x)>1$.

   (h) If $f'(r)$ exists, then $\lim_{x\to r}f(x)=f(r)$.

   (i) The equation $x^{10}-10x^2+5=0$ has a root in the interval $(0,2)$.

   (j) A rational function can have two different horizontal asymptotes.

Solution: (a) F (limit law does not apply to infinite limits)
(b) T (polynomials are continuous)
(c) F (example: $f(x)=x$, $g(x)=x^{-1}$, $a=0$)
(d) F (example: $f(x)=x^{-2}$, $g(x)=-x^{-4}$, $a=0$)
(e) F (limits in the definition of the vertical asymptote "ignore" $f(1)$)
(f) F (example: $f(x)=|x|$)
(g) F (example: $f(x)=x+1$)
(h) T (differentiable implies continous)
(i) T (apply Intermediate Value Theorem to $[0,1]$)

2. [10] Give a simple example or wite "N/A" if there is no such example.

   (a) A polynomial that is a power function.
   (b) A polynomial that is not a rational function.
   (c) A rational function that is not a polynomial.
   (d) An inverse trigonometric function that is not an algebraic function.
   (e) A continuous function without horizontal and vertical asymptotes.
   (f) A function with 3 vertical asymptotes.
   (g) A root function whose domain does not include -1.
   (h) An algebraic function with domain $(-1, 1)$.
   (i) A function with infinitely many discontinuities.
   (j) A continuous but not differentiable function.

Solution:

(a) $f(x) = 1$ $(= x^0)$
(b) N/A (by definition any polynomial is a rational function)
(c) $f(x) = \frac{1}{x}$
(d) $\sin^{-1}$ (essentially any inverse trigonometric function works)
(e) $f(x) = x$
(f) $f(x) = \frac{1}{x^3 - x}$ $(= \frac{1}{(x-(-1))(x-0)(x-1)})$
(g) $f(x) = \sqrt{x}$
(h) $f(x) = 1/\sqrt{1 - x^2}$
(i) $f(x) = [\![x]\!]$ (discontinuous at all integers)
(j) $f(x) = |x|$

3. [10]

   (a) For $f(x) = \frac{2x^2-18}{x^2+2x-3}$, find all asymtotes and the limits that describe the asymptotic behavior of the function.

   (b) Find the horizontal asymptotes of the function $f(x) = \frac{\sqrt{x^6-1}}{x^3+7x^2+4x-8}$.

Solution: (a) Dropping the terms with not highest exponents in the numerator and denominator of f (as explained in the lecture) yields $y = \frac{2x^2}{x^2} = 2$ as horizontal asymptote for both $x \to \infty$ and $x \to -\infty$. So we have $\lim_{x\to\pm\infty} f(x) = 2$. To find the vertical asymptotes, we factorize and cancel factors if possibe:

$$f(x) = \frac{2x^2 - 18}{x^2 + 2x - 3} = 2\frac{(x+3)(x-3)}{(x+3)(x-1)} = 2\frac{x-3}{x-1}$$

So, $x = 1$ is the only vertical asymptote. As $x - 3 < 0$ for x close to 1, we have $\lim_{x\to 1^-} = \infty$ and $\lim_{x\to 1^+} = -\infty$.

(b) Dropping the terms with not highest exponents under the root and in the denominator of f (as above) yields $\frac{\sqrt{x^6}}{x^3} = \frac{|x|^3}{x^3} = |x|/x$ which has the same asymptotic behavior as $f(x)$ for large $|x|$. So $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} |x|/x = 1$ and similarly $\lim_{x\to-\infty} f(x) = -1$. In other words, $y = 1$ and $y = -1$ are two (different) horizontal asymptotes.

4. [10]

(a) Find all values for a and b such that the function

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if } x < 1 \\ (x-a)^2 & \text{if } 1 \leq x < 2 \\ 2ax - b & \text{if } 2 \leq x \end{cases}$$

becomes continuous.

(b) Is f differentiable for some choice of a and b?

Solution:

(a) First, note that $f(x) = x + 3$ for $x < 1$. Continuity is clear at $x \neq 1, 2$. The following two conditions are equivalent to continuity at 1 and 2 respectively:

$$4 = \lim_{x \to 1^-} f(x) = f(1) = (1-a)^2,$$
$$(2-a)^2 = \lim_{x \to 2^-} f(x) = f(2) = 4a - b.$$

The first equality gives $a = 1 \mp 2$, so $a = -1$ or $a = 3$. Then the second equality reads $5 \pm 4 = (1 \pm 2)^2 = 4 \mp 8 - b$ which gives $b = -1 \mp 12$. So either $a = -1$ and $b = -13$ or $a = 3$ and $b = 11$.

(b) Note that if f is continuous, then $f(1) = 4$ by the first part. For $f'(1) = \lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$ to exist, the corresponding left- and right-sided limits

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{1 + h + 3 - 4}{h} = 1,$$
$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1+h-a)^2 - 4}{h}$$

must be equal. But, for $a = 1 \mp 2$, the right-sided limit equals

$$\lim_{h \to 0^+} \frac{(h \pm 2)^2 - 4}{h} = \lim_{h \to 0^+} \frac{h^2 \pm 4h}{h} = \pm 4$$

which is not equal to the left-sided limit. So the answer is “no”.

5. [10]

(a) Compute the derivative of $f(x) = \frac{1-x}{1+x}$ (using the limit definition).

(b) Find the domains of $f(x)$ and $f'(x)$.

Solution:

(a)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1-x-h}{1+x+h} - \frac{1-x}{1+x}}{h} \\
&= \lim_{h\to 0} \frac{(1-x-h)(1+x) - (1-x)(1+x+h)}{h(1+x+h)(1+x)} \\
&= \lim_{h\to 0} \frac{1-x-h+x-x^2-hx-1-x-h+x+x^2+hx}{h(1+x+h)(1+x)} \\
&= \lim_{h\to 0} \frac{-2h}{h(1+x+h)(1+x)} = -\frac{2}{(1+x)^2}
\end{aligned}$$

(b) Both domains are obviously $\mathbb{R} \setminus \{-1\}$.

6. [10] Compute the limits.

(a) $\lim_{x\to-\infty} \frac{\sqrt{4x^6-x}}{x^3+9}$

(b) $\lim_{t\to\infty} \frac{t^2-t}{2t^2+t+7}$

(c) $\lim_{x\to 0} \left(x^4 \cos \frac{2}{x}\right)$ (Hint: use the Squeeze Theorem)

(d) $\lim_{x\to\frac{\pi}{8}} \arctan\left(\frac{64x^2-\pi^2}{64x-8\pi}\right)$

(e) $\lim_{x\to\pi} \sin(x+\sin(x+\sin(x+\sin(x+\sin x))))$

Solution:

(a) Note that for $x < 0$, we have $x^{-3} = -\sqrt{x^{-6}}$. Using this, we compute $\lim_{x\to-\infty} \frac{\sqrt{4x^6-x}}{x^3+9} = \lim_{x\to-\infty} -\frac{\sqrt{4x^6-x}\sqrt{x^{-6}}}{(x^3+9)x^{-3}} = \lim_{x\to-\infty} -\frac{\sqrt{4-\frac{1}{x^5}}}{1+\frac{9}{x^3}} = -2.$

(b) $\lim_{t\to\infty} \frac{t^2-t}{2t^2+t+7} = \lim_{t\to\infty} \frac{t^2}{2t^2} = \frac{1}{2}$

(c) We have $-x^4 \leq x^4 \cos \frac{2}{x} \leq x^4$ and $\lim_{x\to 0} x^4 = 0$. So, by the Squeeze Theorem, it follows that also $\lim_{x\to 0}\left(x^4 \cos\frac{2}{x}\right) = 0$.

(d) By Theorem 8 in Section 2.5, $\lim_{x\to\frac{\pi}{8}} \arctan\left(\frac{64x^2-\pi^2}{64x-8\pi}\right) = \arctan\left(\lim_{x\to\frac{\pi}{8}} \frac{64x^2-\pi^2}{64x-8\pi}\right)$. But $\lim_{x\to\frac{\pi}{8}} \frac{64x^2-\pi^2}{64x-8\pi} = \lim_{x\to\frac{\pi}{8}}(x+\frac{\pi}{8}) = \frac{\pi}{4}$ and hence the result is $\arctan\frac{\pi}{4} = 1$.

(e) By continuity, $\lim_{x\to\pi} \sin(x+\sin(x+\sin(x+\sin(x+\sin x)))) = \sin(\pi+\sin(\pi+\sin(\pi+\sin(\pi+\sin\pi)))) = 0$.

End of examination
Total pages: 7
Total marks: 60