Given name:____________________ **Family name:**____________________

Student number:____________________ **Signature:**__________________________

OKLAHOMA STATE UNIVERSITY
Department of Mathematics

MATH 2163 (Calculus III)
Instructor: Mathias Schulze

MIDTERM 3
November 20, 2006

Duration: 50 minutes

No aids allowed

This examination paper consists of **4** pages and **6** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer **5 of 6** questions.

To obtain credit, you must explicitly state your result and give arguments to support your answer.

For graders' use:

	Score
1 (10)	
2 (10)	
3 (10)	
4 (10)	
5 (10)	
6 (10)	
Total (60)	

1. [10] Write the following double and triple integrals as iterated integrals. As usual r, θ are the polar coordinates corresponding to x, y and ρ, θ, ϕ are the spherical coordinates corresponding to x, y, z.

   (a) $\iint_D f(x,y)\,dA$ where $D = \{(x,y) \mid g(y) \le x \le h(y), a \le y \le b\}$.
   (b) $\iint_E f(x,y)\,dA$ where $E = \{(x,y) \mid \alpha \le \theta \le \beta, u(\theta) \le r \le v(\theta)\}$.
   (c) $\iiint_B f(x,y,z)\,dV$ where $B = [a,b] \times [u,v] \times [r,s]$.
   (d) $\iiint_S f(x,y,z)\,dV$ where $S = \{(x,y,z) \mid (x,y) \in D, g(x,y) \le z \le h(x,y)\}$.
   (e) $\iiint_T f(x,z)\,dV$ where $T = \{(x,y,z) \mid \alpha(\phi,\theta) \le \rho \le \beta(\phi,\theta)\}$.

   Solution:

   (a) $\int_a^b \int_{g(x)}^{h(y)} f(x,y)\,dx\,dy$
   (b) $\int_\alpha^\beta \int_{u(\theta)}^{v(\theta)} f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta$
   (c) $\int_a^b \int_u^v \int_r^s f(x,y,z)\,dz\,dy\,dx$
   (d) $\iint_D \int_{g(x,y)}^{h(x,y)} f(x,y,z)\,dz\,dA$
   (e) $\int_0^{2\pi} \int_0^\pi \int_{\alpha(\phi,\theta)}^{\beta(\phi,\theta)} f(\rho\sin\phi\cos\theta, \rho\cos\phi)\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

2. [10] Use integration to compute the area of a disk with radius R.

   Solution: To compute the area of the disk $D = \{(x,y) \mid r \le R\}$ we integrate the function 1 over D in polar coordinates:

$$A(D) = \iint_D 1\,dA = \int_0^{2\pi} \int_0^R r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^R r\,dr = \frac{2\pi}{2}[r^2]_0^R = \pi R^2$$

   The result is what we expected.

3. [10] Evaluate the integral $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{2-x^2-y^2}(x^2+y^2)^{3/2}\,dz\,dy\,dx$ by changing to cylindrical coordinates.

Solution: Observe that $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ is equivalent to $y^2 \leq 1-x^2$ and hence to $r \leq 1$ in polar coordinates. Te condition $-1 \leq x = r\cos\theta \leq 1$ is therefore redundant. In other words, there is no restriction for θ. Also note that $(x^2+y^2)^{3/2} = r^3$ in polar coordinates. So the given integral equals

$$\int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} r^3\,dz\,r dr\,d\theta = 4\pi\int_0^1 (1-r^2)r^4\,dr = 4\pi\left(\frac{1}{5}-\frac{1}{7}\right) = \frac{8\pi}{35}$$

4. [10] Find the volume V that lies both within the cylinder $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$.

Solution: This has been a homework problem for lecture 35. Here is the solution:

$$\begin{aligned} V &= \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1\,dz\,r dr\,d\theta = \int_0^{2\pi}d\theta\int_0^1 [z]_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}}\cdot r\,dr \\ &= 4\pi\int_0^1 r\sqrt{4-r^2}\,dr = 4\pi\left[-\frac{1}{3}\sqrt{4-r^2}^3\right]_0^1 = -4\pi\sqrt{3} \end{aligned}$$

5. [10] Evaluate $\iiint_B x^2 + y^2 + z^2 \, dV$ where B is the unit ball $x^2 + y^2 + z^2 \leq 1$.

Solution: Note that $x^2 + y^2 + z^2 = \rho^2$ in spherical coordinates and hence

$$\begin{aligned}\iiint_B x^2 + y^2 + z^2 \, dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_0^1 \rho^4 \, d\rho \\ &= 2\pi[-\cos\phi]_0^{\pi} \frac{1}{5} \\ &= \frac{4\pi}{5}\end{aligned}$$

6. [10] Find the mass center of a solid hemisphere H of radius 1 centered at the origin above the x-y-plane where the density equals z.

Solution: This is a simplified version of a homework problem for lecture 36. Here is the solution: In cylindrical coordinates, the density is $z = \rho\cos\phi$. So the mass of the object is given by

$$\begin{aligned}m &= \iiint_H z \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho\cos\phi \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\pi/2} \frac{1}{2}\sin(2\phi) \, d\phi \int_0^1 \rho^3 \, d\rho = 2\pi \left[-\frac{1}{4}\cos(2\phi)\right]_0^{\pi/2} \frac{1}{4} = \frac{\pi}{4}.\end{aligned}$$

For symmetry reasons the mass center has x- and y-coordinate equal to 0. Its z-coordinate is given by the formula

$$\begin{aligned}m \cdot \bar{z} &= \iiint_H z^2 \, dV = \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos^2\phi \sin\phi \, d\phi \int_0^1 \rho^4 \, d\rho \\ &= 2\pi \left[-\frac{1}{3}\cos^3\phi\right]_0^{\pi/2} \frac{1}{5} = \frac{2\pi}{15}.\end{aligned}$$

From the calculation of m above we conclude that $\bar{z} = \frac{1}{m}\frac{2\pi}{15} = \frac{8}{15}$. Thus $\left(0, 0, \frac{8}{15}\right)$ is the mass center of the given hemisphere H.

End of examination
Total pages: 4
Total marks: 60