Given name:____________________ **Family name:**____________________

Student number:____________________ **Signature:**____________________________

OKLAHOMA STATE UNIVERSITY
Department of Mathematics

MATH 2163 (Calculus III)
Instructor: Mathias Schulze

MIDTERM 2
October 23, 2006

Duration: 50 minutes

No aids allowed

This examination paper consists of **4** pages and **6** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer **5 of 6** questions.

To obtain credit, you must explicitly state your result and give arguments to support your answer.

For graders' use:

	Score
1 (0)	
2 (10)	
3 (10)	
4 (10)	
5 (10)	
6 (10)	
Total (60)	

1. [10] For the function $f(x,y) = x+xy+y$ compute: the partial derivatives, the gradient, the directional derivative at $Q(0,0)$ toward the point $P(3,-4)$, and the maximal and minimal rate of change of f at $Q(0,0)$.

   Solution:

   - $f_x = 1+y$, $f_y = 1+x$.
   - $\vec{\nabla} f = \langle 1+y, 1+x \rangle$, $(\vec{\nabla} f)(0,0) = \langle 1, 1 \rangle$.
   - $\vec{u} = \vec{QP} = \frac{\langle 3,-4 \rangle}{\sqrt{9+16}} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$.
   - $(D_{\vec{u}} f)(0,0) = (\vec{\nabla} f)(0,0) \cdot \vec{u} = \langle 1, 1 \rangle \cdot \langle \frac{3}{5}, -\frac{4}{5} \rangle = -\frac{1}{5}$.
   - The maximal/minimal rate of change of f at $Q(0,0)$ is $\pm |(\vec{\nabla} f)(0,0)| = \pm\sqrt{2}$.

2. [10] Compute *all* local minima, maxima, and saddle points of the function $f(x,y) = xy - x - y + 1$

   Solution: If you write $xy - x - y + 1 = (x-1)(y-1)$ and change coordinates $x' = x-1, y' = y-1$ this problem is identical to Quiz 4. Here is the solution again: The first and second order derivatives are

   $$f_x = y-1, \quad f_y = x-1, \quad f_{x,x} = 0, \quad f_{y,y} = 0, \quad f_{x,y} = f_{y,x} = 1.$$

   Setting f_x and f_y equal to zero gives only one critical point, namely $(1,1)$. The Hessian matrix is constant and equals

   $$H_f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

   Thus $D_f = |H_f| = 0 \cdot 0 - 1 \cdot 1 < 0$ and the second derivative test tells us that $(1,1)$ is a saddle point.

   Answer: The *only* critical point $(1,1)$ of f is a saddle point.

3. [10] Find the points (x, y) where the gradients of the two functions $f(x, y) = (x-1)^2+y^2$ and $g(x, y) = (x + 1)^2 + y^2$ are orthogonal. Explain your result geometrically (text or picture).

Solution: The gradients of the two functions are, up to a factor 2, $\vec{\nabla} f \equiv \langle x - 1, y\rangle$ and $\vec{\nabla} g \equiv \langle x + 1, y\rangle$. They are orthogonal exactly if the dot product

$$\vec{\nabla} f \cdot \vec{\nabla} g \equiv (x - 1)(x + 1) + y^2 = x^2 + y^2 - 1$$

equals 0. This is equivalent to the condition $x^2 + y^2 = 1$. *Conclusion:* The points in question are exactly the points on a unit circle centered at the origin.

Geometrical explanation: The level curves of f are circles centered at $(1, 0)$, those of g are circles centered at $(-1, 0)$. Recall that at each point the gradient is normal to the level curve through this point. Thus the two gradients are orthogonal at (x, y) exactly if the level curves of f and g through (x, y) are orthogonal. This means that the triangle with vertices $(-1, 0)$, $(1, 0)$, (x, y) has a $90°$ angle at (x, y). Thales' Theorem states that all (x, y) for which this is true lie on the circle whose diameter is the line segment between $(-1, 0)$ and $(1, 0)$. So the above computation verifies Thales' Theorem.

4. [10] Use the method of Lagrange multipliers to find the maximal volume of a circular cylinder with fixed surface area equal to 1.

Solution: Let $r > 0$ be the radius and $h > 0$ be the height of the cylinder. Then its volume is $V(r, h) = \pi r^2 h$ and its surface is $S(r, h) = 2\pi r^2 + 2\pi rh = 2\pi(r^2 + rh)$. Up to constant factors, the gradients of these two functions are

$$\vec{\nabla} V(r, h) = \langle 2rh, r^2\rangle, \quad \vec{\nabla} S(r, h) = \langle 2r + h, r\rangle.$$

By the method of Lagrange multipliers, we have to solve the system $\vec{\nabla} V(r, h) = \lambda \cdot \vec{\nabla} S(r, h)$, $S(r, h) = 1$ or, more explicitly,

$$2rh = 2\lambda r + \lambda h, \quad r^2 = \lambda r, \quad r^2 + rh = \frac{1}{2\pi}.$$

Because $r > 0$ the second equation gives $r = \lambda > 0$. Substituting into the first yields $2\lambda h = 2\lambda^2 + \lambda h$ and hence $h = 2\lambda$. Finally we substitute this into our third equation to compute $3\lambda^2 = \frac{1}{2\pi}$ and hence $\lambda = \frac{1}{\sqrt{6\pi}}$. Finally, $r = \frac{1}{\sqrt{6\pi}}$, $h = \sqrt{\frac{2}{3\pi}}$, and the maximal volume is $V(r, h) = \frac{\pi}{6\pi}\sqrt{\frac{2}{3\pi}} = \frac{1}{3\sqrt{6\pi}}$.

Conclusion: The maximal volume of a circular cylinder with surface 1 equals $\frac{1}{3\sqrt{6\pi}}$. The height and diameter of this maximal cylinder both equal $\sqrt{\frac{2}{3\pi}}$.

5. [10] Calculate the double integral $\iint_R \frac{x}{y}\, dA$ over the rectangle $R = [0, 2] \times [1, 2]$.

Solution: This is a simplified version of Exercise 16.2.9 which has been a homework problem for lecture 21.

$$\begin{aligned}\iint_R \frac{x}{y} dA &= \int_0^2 \int_1^2 \frac{x}{y} dy dx \\ &= \int_0^2 x[\log(y)]_1^2 dx \\ &= \log(2) \int_0^2 x dx \\ &= \frac{\log(2)}{2}[x^2]_0^2 \\ &= 2\log(2)\end{aligned}$$

6. [10] Compute the double integral $\iint_D x + 2y\, dA$ over the region D bounded by the two parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution: This is Example 1 in Section 16.3 which has been discussed in lecture 22.

End of examination
Total pages: 4
Total marks: 60