Given name:____________________ **Family name:**____________________

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OKLAHOMA STATE UNIVERSITY
Department of Mathematics

MATH 2163 (Calculus III)
Instructor: Mathias Schulze

MIDTERM 1
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Duration: 50 minutes

No aids allowed

This examination paper consists of **4** pages and **6** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer **5 of 6** questions.

To obtain credit, you must give arguments to support your answers.

For graders' use:

	Score
1 (0)	
2 (10)	
3 (10)	
4 (10)	
5 (10)	
6 (10)	
Total (60)	

1. [10] For the vectors $\vec{v} = 4\vec{i} + 3\vec{k}$ and $\vec{w} = \langle 1, 1, 1 \rangle$ (where $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit vectors), compute the following expressions: $|\vec{v}|$, $\vec{v} - 2\vec{w}$, $\vec{v} \cdot \vec{w}$, and $\vec{v} \times \vec{w}$. Are $\vec{v}$ and $\vec{w}$ orthogonal or parallel?

   Solution:

   - $|\vec{v}| = |\langle 4, 0, 3 \rangle| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{16 + 0 + 9} = \sqrt{25} = 5$
   - $\vec{v} - 2\vec{w} = \langle 4, 0, 3 \rangle - 2\langle 1, 1, 1 \rangle = \langle 4 - 2 \cdot 1, 0 - 2 \cdot 1, 3 - 2 \cdot 1 \rangle = \langle 2, -2, 1 \rangle$
   - $\vec{v} \cdot \vec{w} = \langle 4, 0, 3 \rangle \cdot \langle 1, 1, 1 \rangle = 4 \cdot 1 + 0 \cdot 1 + 3 \cdot 1 = 7$
   - $\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 0 & 3 \\ 1 & 1 & 1 \end{vmatrix} = (0 \cdot 1 - 3 \cdot 1)\vec{i} + (3 \cdot 1 - 4 \cdot 1)\vec{j} + (4 \cdot 1 - 0 \cdot 1)\vec{k} = \langle -3, -1, 4 \rangle$
   - $\vec{v}$ and $\vec{w}$ are not orthogonal because $\vec{v} \cdot \vec{w} = 7 \neq 0$.
   - $\vec{v}$ and $\vec{w}$ are not parallel because $\vec{v} \times \vec{w} = \langle -3, -1, 4 \rangle \neq \vec{0}$.

2. [10] Compute one of the angles in the triangle with vertices $A(-1, 0, 1)$, $B(0, \sqrt{3}, 1)$, $C(1, 0, 1)$.

   Solution: We choose to compute the angle at the vertex A, call it α. (Actually all three angles are equal.) The two edges of the triangle that meet in the vertex A are given by the vectors $\vec{v} = \vec{AB} = \langle 1, \sqrt{3}, 0 \rangle$ and $\vec{w} = \vec{AC} = \langle 2, 0, 0 \rangle$ respectively. Then α is the angle between $\vec{v}$ and $\vec{w}$ which can be computed using the formula

   $$\cos\alpha = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|} = \frac{2}{\sqrt{1+3} \cdot 2} = \frac{1}{2}.$$

   From $\cos\alpha = \frac{1}{2}$ we conclude that $\alpha = \frac{\pi}{3} = 60^\circ$.

3. [10] What are the geometrical interpretations of the following expressions?

   (a) $\frac{\vec{a}}{|\vec{a}|}$

   (b) $|\vec{a} \times \vec{b}|$

   (c) $|\vec{a} \cdot (\vec{b} \times \vec{c})|$

   (d) $\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a}$

   (e) $\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$

   Write 5 short sentences like *(b) is the height of the triangle with edges $\vec{a}$ and $\vec{b}$.*

   Solution:

   (a) is the unit vector (vector of length 1) in direction of $\vec{a}$.
   (b) is the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$.
   (c) is the volume of the parallelepiped spanned by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.
   (d) is the (vector) projection of $\vec{b}$ to the line along the vector $\vec{a}$.
   (e) is the distance between $P(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$.

4. [10] Determine the equation of the plane perpendicular to $\vec{n} = \langle 0, 4, 3 \rangle$ which contains the point $P(3, 2, 1)$. What is the distance of the point $Q(0, 1, 0)$ from this plane?

   Solution: The general (linear) equation of a plane is $ax + by + cz = d$ or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where $\langle a, b, c \rangle$ is the normal vector. For the second equation it is obvious that the point $P(x_0, y_0, z_0)$ is in the plane because substituting $x = x_0$, $y = y_0$, $z = z_0$ yields 0 on the left hand side.
In our case $\vec{n} = \langle 0, 4, 3 \rangle$ is the normal vector and $x_0 = 3$, $y_0 = 2$, $z_0 = 1$ are the coordiantes of a point in the plane. So the equation of the plane is $0(x - 3) + 4(y - 2) + 3(z - 1) = 0$ or equivalently

$$4y + 3z = 4 \cdot 2 + 3 \cdot 1 = 11.$$

Note that $a = 0$, $b = 4$, $c = 3$, and $d = 11$. Denote the coordinates of Q by $x_1 = 0$, $y_1 = 1$, $z_1 = 0$. Then the distance of Q from this plane is

$$D = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|4 \cdot 1 - 11|}{\sqrt{4^2 + 3^2}} = \frac{7}{5}.$$

5. [10] Determine the domain of the function $f(x, y) = \sqrt{x^2 - y}$ and compute the linearization of $f(x, y)$ at $\langle 3, 5\rangle$.

Solution: The only constraint for the domain is that the expression under the root must be positive. This means $x^2 - y \geq 0$ or equivalently $y \leq x^2$ and hence the domain of f is

$$D = \{(x, y) \in \mathbb{R}^2 \mid y \leq x^2\}.$$

The partial derivatives of f are

$$f_x(x, y) = \frac{x}{\sqrt{x^2 - y}}, \quad f_y(x, y) = -\frac{1}{2\sqrt{x^2 - y}}.$$

Note that at $\langle 3, 5\rangle$, $\sqrt{x^2 - 1} = \sqrt{9 - 5} = 2$. So the linearization of f at $\langle 3, 5\rangle$ is

$$\begin{aligned} L(x, y) &= f(3, 5) + f_x(3, 5)(x - 3) + f_y(3, 5)(y - 5) \\ &= 2 + \frac{3}{2}(x - 3) - \frac{5}{4}(y - 5) \\ &= \frac{8 - 18 + 25}{4} + \frac{3}{2}x - \frac{5}{4}y \\ &= \frac{3x}{2} + \frac{5y}{4} + \frac{15}{4} \end{aligned}$$

6. [10] Compute the total differential of $f(x, y) = e^{xy}$ and use it to estimate $f(1.1, 1.1)$.

Solution: The partial derivatives of f are $f_x(x, y) = ye^{xy}$ and $f_y(x, y) = xe^{xy}$. So the total differential of f is

$$df = dz = e^{xy}(ydx + xdy)$$

To estimate $f(1.1, 1.1)$ we choose $x = y = 1$ because $\langle 1, 1\rangle$ is close to $\langle 1.1, 1.1\rangle$ and we can easily compute $e^{1 \cdot 1} = e$. Then $dx = 1.1 - x = 0.1$ and $dy = 1.1 - y = 0.1$ and hence

$$dz = e^{1 \cdot 1}(1 \cdot 0.1 + 1 \cdot 0.1) = 0.2 \cdot e.$$

We approximate $dz \approx \Delta z = f(x + dx, y + dy) - f(x, y) = f(1.1, 1.1) - f(1, 1)$ and conclude that

$$f(1.1, 1.1) \approx f(1, 1) + dz = e + 0.2 \cdot e = 1.2 \cdot e.$$

End of examination
Total pages: 4
Total marks: 60