ON THE TRIANGULATED STRUCTURE
OF STABLE MONOMORPHISM CATEGORIES

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Abstract. We investigate the triangulated structure of stable monomorphism categories (filtered chain categories) over a Frobenius category. The high degree of symmetry of linear quivers leads to a plethora of semiorthogonal decompositions into smaller categories of the same type. These form polygons of recollements, in which a full turn of mutations is a power of a particular auto-equivalence of the stable monomorphism category. A certain power of this auto-equivalence is the square of the suspension functor. We describe the infinite chains of adjoint pairs obtained from the polygons. As an application, we explicate the construction of Bondal and Kapranov for lifting representing objects of dualized hom-functors in our setup.

Contents

1. Introduction

A celebrated result of Buchweitz's gives a triangle equivalence between the singularity category of a Gorenstein ring R and the stable category of maximal Cohen-Macaulay, that is, Gorenstein projective R-modules, see [Buc21, Thm. 4.4.1]. Over a hypersurface ring $R = S/\langle f\rangle$, the latter category is triangle equivalent to the homotopy category of matrix factorizations of f (with two factors) due to Eisenbud's Theorem, see [Yos90, Thm. 7.4].

In [FS24, Thm. 3.37] and [BM24, Thms. 4.12, 5.3], Buchweitz's theorem was generalized, replacing

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the category of Gorenstein projective modules by the monomorphism category $\operatorname{Mor}^{\mathrm{m}}_{l}(\mathcal{F})$, consisting of chains

$$X = (X, \alpha)\colon\ X^0 \overset{\alpha^0}{\rightarrowtail} X^1 \overset{\alpha^1}{\rightarrowtail} \cdots \overset{\alpha^{l-1}}{\rightarrowtail} X^l.$$

of l admissible monomorphisms over a Frobenius subcategory $\mathcal{F}$ of an exact category $\mathcal{E}$. This is a fully exact subcategory of the category $\operatorname{Mor}_l(\mathcal{E})$ of chains of l morphisms in $\mathcal{E}$. For the subcategory $\mathcal{F}$ of Gorenstein projectives of suitable module categories $\mathcal{E}$, $\operatorname{Mor}^{\mathrm{m}}_{l}(\mathcal{F})$ is the subcategory of Gorenstein projectives of $\operatorname{Mor}_l(\mathcal{E})$, see [JK11, Cor. 3.6] and [SZ24, Thm. 4.1]. The triangle equivalence with matrix factorizations generalizes to factorizations with $l + 2$ factors, see [SZ24, Thm. 4.6]. Notably, the (homotopy) category of such factorizations has a cyclic symmetry.

In this article, we investigate the triangulated structure of the above mentioned stable category $\mathcal{M}_l := \underline{\operatorname{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$. Our approach is inspired by the work of Iyama, Kato, and Miyachi on the homotopy category of complexes over split monomorphism categories, see [IKM16, §4]. We consider two operations relating chains of monomorphisms of different lengths: By *contraction* of an interval $[s, t] \subseteq \{0, \ldots, l\}$, a chain $X = (X, \alpha) \in \operatorname{Mor}^{\mathrm{m}}_{l}(\mathcal{F})$ as above is sent to

$$\gamma^{[s,t]}(X)\colon\ X^0 \overset{\alpha^0}{\rightarrowtail} \cdots \overset{\alpha^{s-2}}{\rightarrowtail} X^{s-1} \overset{\alpha^t\cdots\alpha^{s-1}}{\rightarrowtail} X^{t+1} \overset{\alpha^{t+1}}{\rightarrowtail} \cdots \overset{\alpha^{l-1}}{\rightarrowtail} X^l \in \operatorname{Mor}^{\mathrm{m}}_{l-t+s-1}(\mathcal{F}),$$

by *expansion* with identities to

$$\delta^{[s,t]}(X)\colon\ X^0 \overset{\alpha^0}{\rightarrowtail} \cdots \overset{\alpha^{s-1}}{\rightarrowtail} X^s = \cdots = X^s \overset{\alpha^s}{\rightarrowtail} \cdots \overset{\alpha^{l-1}}{\rightarrowtail} X^l \in \operatorname{Mor}^{\mathrm{m}}_{l+t-s}(\mathcal{F}).$$

Both assignments give rise to exact functors inducing triangulated functors $\underline{\gamma}^{[s,t]}$ and $\underline{\delta}^{[s,t]}$ between the respective stable categories. The kernel $\Gamma^{[s,t]}$ of $\underline{\gamma}^{[s,t]}$ and the image $\Delta^{[s,t]}$ of $\underline{\delta}^{[s,t]}$ (with l replaced by $l - t + s$) are triangulated subcategories of $\underline{\operatorname{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$. These occur in various *semiorthogonal decompositions*:

Theorem A. *The category* $\underline{\operatorname{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$ *admits the following semiorthogonal decompositions:*

(a) $\left(\Gamma^{[s+1,l]}, \Gamma^{[0,s]}\right)$ *if* $s < l$,

(b) $\left(\Gamma^{[s,t]}, \Delta^{[s,t+1]}\right)$ *if* $t < l$,

(c) $\left(\Delta^{[s-1,t]}, \Gamma^{[s,t]}\right)$ *if* $s > 0$.

From these decompositions, we derive further results. Two semiorthogonal decompositions $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{W})$ define a *recollement*, see [IKM16, Prop. 1.2]. As a direct consequence, we find a family of *polygons of recollements*:

Theorem B. *There are the following* $(2l + 4)$*-gons of recollements in* $\underline{\operatorname{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$ *for* $s < l$*:*

$$\begin{tikzcd}
 & \Delta^{[0,s+1]} \arrow[r, tail] & \cdots \arrow[r, tail] & \Gamma^{[k,k+s]} \arrow[r, tail] & \Delta^{[k,k+s+1]} \arrow[r, tail] & \Gamma^{[k+1,k+s+1]} \arrow[r, tail] & \cdots \arrow[r, tail] & \Gamma^{[l-s,l]} \arrow[dr, tail, bend left] & \\
\Gamma^{[0,s]} \arrow[ur, tail, bend left] & & & & & & & & \Gamma^{[0,l-s-1]} \arrow[dl, tail, bend left] \\
 & \Gamma^{[s+1,l]} \arrow[ul, tail, bend left] & \cdots \arrow[l, tail] & \Delta^{[k+1,k+l-s+1]} \arrow[l, tail] & \Gamma^{[k+1,k+l-s]} \arrow[l, tail] & \Delta^{[k,k+l-s]} \arrow[l, tail] & \cdots \arrow[l, tail] & \Delta^{[0,l-s]} \arrow[l, tail] &
\end{tikzcd}$$

We describe the *left mutations* L defined by the recollements in these polygons explicitly in terms of an auto-equivalence Θ of $\underline{\operatorname{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$, see Construction 5.3, which corresponds to the rotation functor for matrix factorizations with $l+2$ factors. The result can be visualized by two nested polygons of triangle equivalences, see Figure 1, where $\underline{\delta}^{[s,t]^c}$ is another type of expansion functor with image $\Gamma^{[s,t]}$, see Construction 5.2.

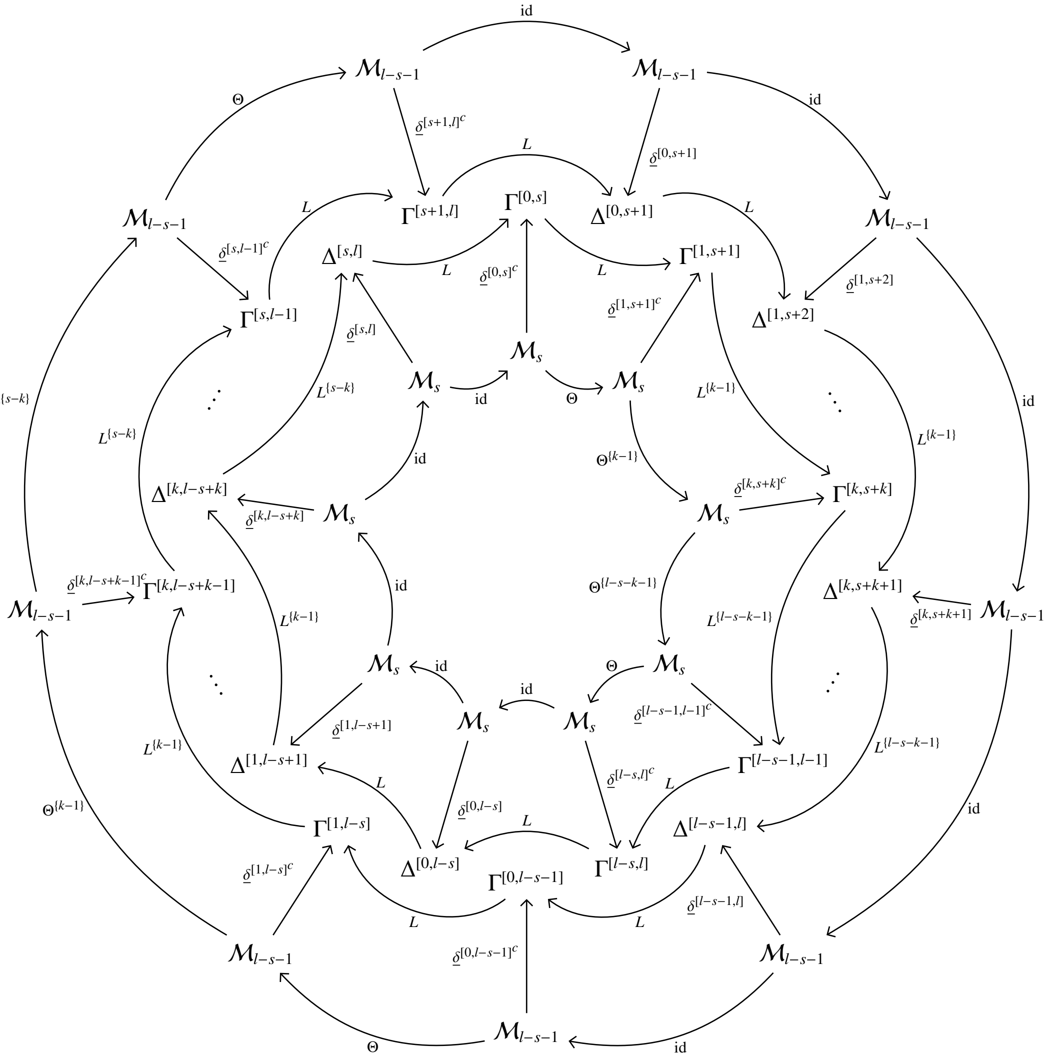

FIGURE 1. Polygons of recollements and mutations

A full turn of left mutations in each such polygons corresponds to a certain power of Θ. We relate such powers to the suspension functor Σ of $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$, which shows, in particular, that $\Sigma^2 \cong \mathrm{id}$ for matrix factorizations with $l+2$ factors, see [Tri21, Prop. 5.3]:

Theorem C. *There is an isomorphism* $\Theta^{l+2} \cong \Sigma^2$ *of endofunctors on* $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$.

There is a triangle equivalence between $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ and the homotopy category of acyclic $(l+2)$-complexes over $\mathcal{F}$ with projective objects, see [FS24, Thm. 2.50], under which Θ corresponds to the shift functor and Theorem C to [IKM17, Thm. 2.4].

We include two further types of contraction functors $\hat{\underline{\gamma}}^{[s,t]^c}$ and $\check{\underline{\gamma}}^{[s,t]^c}$, see Construction 6.2, to form *infinite adjoint chains*:

Theorem D. *There are the following infinite adjoint chains:*

$$\cdots \dashv \Theta^2 \hat{\underline{\gamma}}^{[t-s-1,l-2]^c} \dashv \underline{\delta}^{[t-s-1,l-2]^c}\Theta^{-2} \dashv \Theta\hat{\underline{\gamma}}^{[t-s,l-1]^c} \dashv \underline{\delta}^{[t-s,l-1]^c}\Theta^{-1} \dashv \hat{\underline{\gamma}}^{[t-s+1,l]^c} \dashv \underline{\delta}^{[t-s+1,l]^c}$$
$$\dashv \underline{\gamma}^{[0,t-s]} \dashv \cdots \dashv \underline{\gamma}^{[s-1,t-1]} \dashv \underline{\delta}^{[s-1,t]} \dashv \underline{\gamma}^{[s,t]} \dashv \underline{\delta}^{[s,t+1]} \dashv \underline{\gamma}^{[s+1,t+1]} \dashv \cdots \dashv \underline{\gamma}^{[l-t+s,l]}$$
$$\dashv \underline{\delta}^{[0,l-t+s-1]^c} \dashv \check{\underline{\gamma}}^{[0,l-t+s-1]^c} \dashv \underline{\delta}^{[1,l-t+s]^c}\Theta \dashv \Theta^{-1}\check{\underline{\gamma}}^{[1,l-t+s]^c} \dashv \underline{\delta}^{[2,l-t+s+1]^c}\Theta^2 \dashv \Theta^{-2}\check{\underline{\gamma}}^{[2,l-t+s+1]^c} \dashv \cdots$$

As an application of our results, we explicitly describe *representing objects* of *dualized hom-functors* on $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ lifted from $\underline{\mathcal{F}}$ using the construction of Bondal and Kapranov:

Theorem E. *Let $\mathcal{F}$ be a Frobenius category, linear over a field, such that $\underline{\mathcal{F}}$ is finite-finite. Suppose that the dualized hom-functors* $\mathrm{Hom}(A,-)^*\colon \underline{\mathcal{F}} \to \mathrm{Vect}$ *for all $A \in \mathcal{F}$ are representable. Then the dualized hom-functor* $\mathrm{Hom}(X,-)^*\colon \underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F}) \to \mathrm{Vect}$ *for any $X = (X,\alpha) \in \mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$ is representable by an object, obtained as the rightmost column of any diagram*

$$\begin{tikzcd}
\tilde{X}^{0,0} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \tilde{X}^{0,1} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \tilde{X}^{0,2} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \cdots \arrow[r] \arrow[dr, phantom, "\square"] & \tilde{X}^{0,l-2} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \tilde{X}^{0,l-1} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \tilde{X}^{0,l} \arrow[d, tail] \\
I^0 \arrow[r] & \tilde{X}^{1,1} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \tilde{X}^{1,2} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \cdots \arrow[r] \arrow[dr, phantom, "\square"] & \tilde{X}^{1,l-2} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \tilde{X}^{1,l-1} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \tilde{X}^{1,l} \arrow[d, tail] \\
 & I^1 \arrow[r] & \tilde{X}^{2,2} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \cdots \arrow[r] & \tilde{X}^{2,l-2} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \tilde{X}^{2,l-1} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \tilde{X}^{2,l} \arrow[d, tail] \\
 & & I^2 \arrow[r] & \cdots & \vdots \arrow[d, tail] \arrow[dr, phantom, "\square"] & \vdots \arrow[d, tail] \arrow[dr, phantom, "\square"] & \vdots \arrow[d, tail] \\
 & & & & I^{l-2} \arrow[r] & \tilde{X}^{l-1,l-1} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \tilde{X}^{l-1,l} \arrow[d, tail] \\
 & & & & & I^{l-1} \arrow[r] & \tilde{X}^{l,l},
\end{tikzcd}$$

of bicartesian squares in $\mathcal{F}$ with $I^j \in \mathrm{Inj}(\mathcal{F})$, where the first row consists of the representing objects $\tilde{X}^{0,j}$ of $\mathrm{Hom}(X^j,-)^*$ *and the morphisms corresponding to α^j.*

In particular, if $\underline{\mathcal{F}}$ admits a Serre functor sending the diagram in $\underline{\mathcal{F}}$ defined by any $X \in \mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$ to the one defined by the first row of the diagram in Theorem E, then $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ admits a Serre functor, sending X to $\tilde{X}$, see [BK89, Prop. 3.4.(a)]. It remains to describe this Serre functor on morphisms.

Theorem A summarizes Propositions 4.1 and 4.4 of the main part. Theorems B to D correspond to Corollary 4.7, Proposition 5.5, and Theorem 6.5. Theorem E joins Theorem 7.7 and Proposition 7.8.

2. Triangulated categories of monomorphisms

In this section, we review preliminaries on monomorphism categories in the context of exact and triangulated categories.

Unless stated otherwise, all (sub)categories and functors considered are assumed to be (full) additive. Our main reference on the topic of *triangulated categories* is Neeman's book [Nee01]. However, we require the more general definition of a triangulated category whose suspension functor is only an auto-equivalence instead of an automorphism. These two definitions agree up to a triangulated equivalence, see [KV87, §2] and [May01, §2].
Recall that a *triangle equivalence* is a triangulated functor which is an equivalence of categories. Its quasi-inverse is automatically a triangulated functor, see [BK89, Prop. 1.4] for a more general statement.

Definition 2.1. A pair $(\mathcal{U}, \mathcal{V})$ of triangulated subcategories of a triangulated category $\mathcal{T}$ is called a **semiorthogonal decomposition** of $\mathcal{T}$ if

(a) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$ and

(b) each $T \in \mathcal{T}$ fits into a distinguished triangle $U \to T \to V$, where $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Remark 2.2. Consider two triangulated subcategories $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{T}$ with $\mathrm{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$. Then any solid diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d, dashed] & X \arrow[r] \arrow[d, "f"] & V \arrow[d, dashed] \\
U' \arrow[r] & X' \arrow[r] & V'.
\end{tikzcd}$$

in $\mathcal{T}$ with $U, U' \in \mathcal{U}$ and $V, V' \in \mathcal{V}$ whose rows are distinguished triangles extends uniquely by dashed arrows to a commutative diagram.

This leads to the following

Proposition 2.3 ([IKM11, Prop. 1.2])**.** *Let $(\mathcal{U}, \mathcal{V})$ be a semiorthogonal decomposition of a triangulated category $\mathcal{T}$. Then the inclusion functors $i_! \colon \mathcal{U} \to \mathcal{T}$ and $j_* \colon \mathcal{V} \to \mathcal{T}$ have a respective right adjoint $i^! \colon \mathcal{T} \to \mathcal{U}$ and left adjoint $j^* \colon \mathcal{T} \to \mathcal{V}$. They are given by fixing for each $X \in \mathcal{T}$ a distinguished triangle*

$$i_! i^! X \longrightarrow X \longrightarrow j_* j^* X$$

in $\mathcal{T}$, whose morphisms are then given by the respective unit and counit. These adjoints induce triangle equivalences $\mathcal{T}/\mathcal{V} \to \mathcal{U}$ and $\mathcal{T}/\mathcal{U} \to \mathcal{V}$, quasi-inverse to the composed canonical functors $\mathcal{U} \hookrightarrow \mathcal{T} \to \mathcal{T}/\mathcal{V}$ and $\mathcal{V} \hookrightarrow \mathcal{T} \to \mathcal{T}/\mathcal{U}$, respectively. □

Notation 2.4. Adjoint functors $L \dashv R$ between categories $\mathcal{C}$ and $\mathcal{D}$ will be displayed as

$$\begin{tikzcd}
\mathcal{D} \arrow[r, bend right, "R"'] & \mathcal{C}, \arrow[l, bend right, "L"']
\end{tikzcd}$$

where the left adjoint L is always the upper arrow, the right adjoint R the lower arrow.

Definition 2.5. Due to Proposition 2.3, two semiorthogonal decompositions $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{W})$ of a triangulated category $\mathcal{T}$ patch together to form a **recollement**, see [IKM17, Prop. 1.2],

$$\begin{tikzcd}
 & & \mathcal{U} \arrow[dl, bend left] & \\
\mathcal{V} \arrow[r, hook] & \mathcal{T} \arrow[l, bend right] \arrow[l, bend left] \arrow[ur, bend right] \arrow[dr, bend left] \arrow[rr, two heads] & & \mathcal{T}/\mathcal{V} \arrow[ul, "\simeq"'] \arrow[dl, "\simeq"] \\
 & & \mathcal{W}. \arrow[ul, bend right] &
\end{tikzcd}$$

The composed triangle equivalence

$$L := L_{\mathcal{V}} \colon \mathcal{U} \hookrightarrow \mathcal{T} \to \mathcal{W} =: L_{\mathcal{V}}(\mathcal{U}) =: L(\mathcal{U})$$

is the **left mutation** of $\mathcal{U}$ through $\mathcal{V}$, its quasi-inverse

$$R := R_{\mathcal{V}} \colon \mathcal{W} \hookrightarrow \mathcal{T} \to \mathcal{U} =: R_{\mathcal{V}}(\mathcal{W}) =: R(\mathcal{W})$$

the **right mutation** of $\mathcal{W}$ through $\mathcal{V}$.

Remark 2.6. In Definition 2.5, the composition $\mathcal{U} \xrightarrow{L} \mathcal{W} \hookrightarrow \mathcal{T}$ is right adjoint to $\mathcal{T} \to \mathcal{U}$, the composition $\mathcal{W} \xrightarrow{R} \mathcal{U} \hookrightarrow \mathcal{T}$ is left adjoint to $\mathcal{T} \to \mathcal{W}$.

Definition 2.7 ([IKM16, Def. 1.3])**.** An ***n*-gon of recollements** of a triangulated category $\mathcal{T}$ consists of $n \geq 2$ triangulated subcategories $\mathcal{U}_1, \dots, \mathcal{U}_n$ such that $(\mathcal{U}_i, \mathcal{U}_{i+1})$ is a semiorthogonal decomposition for all $i \in \mathbb{Z}/n\mathbb{Z}$.

Prominent examples of triangulated categories, dubbed *algebraic* by Keller [Kel06, §3.6], are the stable categories of Frobenius (exact) categories, see [Hap88, §2]. Our main reference on the topic of *exact categories* is Bühler's expository article [Büh10]. We review selected material on these types of categories. Admissible monics and epics are represented by $\rightarrowtail$ and $\twoheadrightarrow$, respectively.

Proposition 2.8 ([Büh10, Prop. 2.9])**.** *In an exact category, finite direct sums of short exact sequences are again short exact. In particular, any split short exact sequence is short exact.* □

Proposition 2.9 ([Büh10, Prop. 2.12])**.**

(a) For a square

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[d, "f'"] \\
A' \arrow[r, tail, "i'"] & B'
\end{tikzcd}$$

in an exact category, the following statements are equivalent:

(1) The square is a pushout.

(2) The square is bicartesian.

(3) *The sequence* $\begin{tikzcd} A \arrow[r, tail, "{\begin{pmatrix} i \\ -f \end{pmatrix}}"] & B \oplus A' \arrow[r, two heads, "{\begin{pmatrix} f' & i' \end{pmatrix}}"] & B' \end{tikzcd}$ *is short exact.*

(4) *The square is part of a commutative diagram*

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads] \arrow[d, "f'"] & C \arrow[d, equal] \\
A' \arrow[r, tail, "i'"] & B' \arrow[r, two heads] & C.
\end{tikzcd}$$

(b) *For a square*

$$\begin{tikzcd}
A \arrow[r, two heads, "p'"] \arrow[d, "g'"] & B \arrow[d, "g"] \\
A' \arrow[r, two heads, "p"] & B'
\end{tikzcd}$$

in an exact category, the following statements are equivalent:

(1) *The square is a pullback.*

(2) *The square is bicartesian.*

(3) *The sequence* $\begin{tikzcd} A \arrow[r, tail, "{\begin{pmatrix} p' \\ g' \end{pmatrix}}"] & B \oplus A' \arrow[r, two heads, "{\begin{pmatrix} -g & p \end{pmatrix}}"] & B' \end{tikzcd}$ *is short exact.*

(4) *The square is part of a commutative diagram*

$$\begin{tikzcd}
K \arrow[r, tail] \arrow[d, equal] & A \arrow[r, two heads, "p'"] \arrow[d, "g'"] & B \arrow[d, "g"] \\
K \arrow[r, tail] & A' \arrow[r, two heads, "p"] & B'.
\end{tikzcd}$$

□

Lemma 2.10 (Noether lemma, [Büh10, Ex. 3.7])**.** *Any solid commutative diagram*

$$\begin{tikzcd}
A' \arrow[r, tail] \arrow[d, tail] & B' \arrow[r, two heads] \arrow[d, tail] & C' \arrow[d, tail, dashed] \\
A \arrow[r, tail] \arrow[d, two heads] & B \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, two heads, dashed] \\
A'' \arrow[r, tail] & B'' \arrow[r, two heads] & C''
\end{tikzcd}$$

in an exact category with short exact rows and columns can be uniquely completed by a short exact sequence $C' \rightarrowtail C \twoheadrightarrow C''$*.* □

Definition 2.11. A subcategory $\mathcal{E}' \subseteq \mathcal{E}$ of an exact category is called **(fully) exact** if it is an exact category itself, and if the inclusion functor preserves (and reflects) short exact sequences.[1]

Proposition 2.12 ([Büh10, Ex. 13.5, Prop. 11.3, Cor. 11.4])**.** *The subcategories* $\operatorname{Proj}(\mathcal{E})$ *of projective objects and* $\operatorname{Inj}(\mathcal{E})$ *of injective objects are fully exact with the split exact structure, where* $\operatorname{Proj}(\mathcal{E})$ *is closed under kernels of admissible epics and* $\operatorname{Inj}(\mathcal{E})$ *under cokernels of admissible monics.*

Convention 2.13. Let $\mathcal{E}$ be an exact category.

(a) If $\mathcal{E}$ has enough injectives, we fix for any $A \in \mathcal{E}$ an admissible monic $i_A\colon A \rightarrowtail I(A)$ with $I(A) \in \operatorname{Inj}(\mathcal{E})$ and a cokernel ΣA of i_A. We choose $i_A = \mathrm{id}_A$ whenever $A \in \operatorname{Inj}(\mathcal{E})$.

(b) If $\mathcal{E}$ has enough projectives, we fix for any $A \in \mathcal{E}$ an admissible epic $p_A\colon P(A) \twoheadrightarrow A$ with $P(A) \in \operatorname{Proj}(\mathcal{E})$ and a kernel $\Sigma^{-1}A$ of p_A. We choose $p_A = \mathrm{id}_A$ whenever $A \in \operatorname{Proj}(\mathcal{E})$.

Construction 2.14. Let $\mathcal{F}$ be a Frobenius category. Then Σ becomes an endo-functor of the stable category $\underline{\mathcal{F}}$, defined on representatives of morphisms by a commutative diagram

$$\begin{tikzcd}
X \arrow[r, tail, "i_X"] \arrow[d] & I(X) \arrow[r, two heads] \arrow[d, dashed] & \Sigma X \arrow[d, dashed] \\
Y \arrow[r, tail, "i_Y"] & I(Y) \arrow[r, two heads] & \Sigma Y
\end{tikzcd}$$

in $\mathcal{F}$, independent of choices up to isomorphism of functors, see [Hap88, Rem. 2.2]. The dual diagram makes Σ^{-1} an endo-functor of $\underline{\mathcal{F}}$. For any morphism $f\colon X \to Y$ in $\mathcal{F}$, Proposition 2.9.(a) yields a pushout diagram

$$\begin{tikzcd}
X \arrow[r, tail, "i"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & I(X) \arrow[r, two heads] \arrow[d, "g"] & \Sigma X \arrow[d, equal] \\
Y \arrow[r, tail, "j"] & C(f) \arrow[r, two heads] & \Sigma X
\end{tikzcd}$$

and a short exact sequence

$$\begin{tikzcd}
X \arrow[r, tail, "\begin{pmatrix} i \\ -f \end{pmatrix}"] & I(X) \oplus Y \arrow[r, two heads, "\begin{pmatrix} g & j \end{pmatrix}"] & C(f).
\end{tikzcd}$$

The object $C(f) \in \mathcal{F}$ is called a **cone** of f and the sequence

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r] & C(f) \arrow[r] & \Sigma X.
\end{tikzcd}$$

in $\underline{\mathcal{F}}$ a **standard triangle**. Dually, a **cocone** $C^*(f)$ of f fits into a pullback diagram

$$\begin{tikzcd}
\Sigma^{-1} Y \arrow[r, tail] \arrow[d, equal] & C^*(f) \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
\Sigma^{-1} Y \arrow[r, tail] & P(Y) \arrow[r, two heads, "p"] & Y
\end{tikzcd}$$

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[1] Bühler uses the term *fully exact* for the stronger notion of extension-closedness, see [Büh10, Lem. 10.20].

and into a short exact sequence

$$C^*(f) \rightarrowtail X \oplus P(Y) \xrightarrow{\begin{pmatrix} -f & p \end{pmatrix}} \mathrel{\mkern-14mu}\twoheadrightarrow Y. \tag{2.1}$$

Theorem 2.15 ([Hap88, Rem. 2.2, Thm. 2.6])**.** *The stable category of a Frobenius category is triangulated. The suspension functor is* Σ *with quasi-inverse* Σ^{-1}*, the distinguished triangles are the candidate triangles isomorphic to standard triangles, see Construction 2.14.* □

Definition 2.16. We call a morphism in a Frobenius category $\mathcal{F}$ a **quasi-isomorphism** if it is an isomorphism in the stable category $\underline{\mathcal{F}}$.

Lemma 2.17 ([Hap88, Lem. 2.7])**.** *Any short exact sequence* $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ *in a Frobenius category* $\mathcal{F}$ *induces a distinguished triangle* $X \xrightarrow{i} Y \xrightarrow{p} Z$ *in* $\underline{\mathcal{F}}$*. In particular, any morphism* f *in* $\mathcal{F}$ *yields a distinguished triangle* $C^*(f) \longrightarrow X \xrightarrow{f} Y$ *in* $\underline{\mathcal{F}}$*, see* (2.1)*.* □

Proposition 2.18 ([IKM16, Prop. 7.3])**.** *Any exact functor* $F\colon \mathcal{F}' \to \mathcal{F}$ *of Frobenius categories preserving projective-injectives induces a triangulated functor* $\underline{F}\colon \underline{\mathcal{F}'} \to \underline{\mathcal{F}}$ *of the respective stable categories.* □

We further specialize to monomorphism categories, which are our objects of study.

Notation 2.19 ([BM24, Def. 3.1], [IKM17, Def 4.1])**.** Let $\mathcal{E}$ be an exact category and $l \in \mathbb{N}$.

(a) Let $\mathrm{Mor}_l(\mathcal{E})$ denote the category of diagrams of Dynkin type A_{l+1} in $\mathcal{E}$, where A_{l+1} is the linear quiver

$$\underset{0}{\bullet} \longrightarrow \underset{1}{\bullet} \longrightarrow \cdots \longrightarrow \underset{l-1}{\bullet} \longrightarrow \underset{l}{\bullet}$$

with l arrows. We denote elements of $\mathrm{Mor}_l(\mathcal{E})$ by

$$X = (X, \alpha)\colon\ X^0 \xrightarrow{\alpha^0} X^1 \xrightarrow{\alpha^1} \cdots \xrightarrow{\alpha^{l-1}} X^l.$$

By $\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{E})$ and $\mathrm{Mor}_l^{\mathrm{sm}}(\mathcal{E})$ we denote the subcategories of $\mathrm{Mor}_l(\mathcal{E})$, where all arrows are admissible or split monics, respectively.

(b) Given an object $A \in \mathcal{E}$ and $n \in \{1, \dots, l+1\}$, we define the object

$$\mu_n(A)\colon\ 0 = \!= \cdots = \!= 0 \rightarrowtail A^{l-n+1} = \!= \cdots = \!= A^l$$

of $\mathrm{Mor}_l^{\mathrm{sm}}(\mathcal{E})$ by $A^k := A$ for $k \in \{l-n+1, \dots, l\}$.

Theorem 2.20 ([BM24, Props. 3.5, 3.9, 3.11, Thm. 3.12])**.** *Let* $\mathcal{E}$ *be an exact category.*

(a) The category $\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{E})$ *is exact with the termwise exact structure.*

(b) We have $\mathrm{Proj}(\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{E})) = \mathrm{Mor}_l^{\mathrm{sm}}(\mathrm{Proj}(\mathcal{E}))$ *and* $\mathrm{Inj}(\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{E})) = \mathrm{Mor}_l^{\mathrm{sm}}(\mathrm{Inj}(\mathcal{E})) = \mathrm{Mor}_l^{\mathrm{m}}(\mathrm{Inj}(\mathcal{E}))$*.*

(c) If $\mathcal{E}$ *has enough projectives resp. injectives, then so has* $\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{E})$*.*

In particular, if $\mathcal{F}$ *is a Frobenius category, then so is* $\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{F})$*, and* $\mathrm{Proj}(\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{F})) = \mathrm{Mor}_l^{\mathrm{m}}(\mathrm{Proj}(\mathcal{F}))$*.* □

For later reference, we construct sections that witness the injectivity of objects of $\mathrm{Mor}_l^{\mathrm{m}}(\mathrm{Inj}(\mathcal{E}))$:

Lemma 2.21. *Let $\mathcal{E}$ be an exact category, $l \in \mathbb{N}$, and $I \in \mathrm{Mor}_l^{\mathrm{m}}(\mathrm{Inj}(\mathcal{E}))$. For an admissible monic $s\colon I \rightarrowtail X$ in $\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{E})$, any left-inverse $r^0\colon X^0 \to I^0$ of s^0 extends to a left-inverse $r\colon X \to I$ of s.*

Proof. Write $I = (I, \iota)$ and $X = (X, \alpha)$. We may assume that $l = 1$. Due to Lemma 2.10, the admissible monic s gives rise to a commutative diagram

$$\begin{tikzcd}
I^0 \arrow[r, tail, "\iota^0"] \arrow[d, tail, "s^0"] & I^1 \arrow[r, two heads, "\kappa^1"] \arrow[d, tail, "s^1"] \arrow[l, dotted, bend right, "\kappa^0"'] & J^1 \arrow[d, tail, dashed, "j^1"] \arrow[l, dotted, bend right, "\iota^1"'] \\
X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, two heads] \arrow[u, bend left, "r^0"] & X^1 \arrow[r, two heads, "\beta^1"] \arrow[d, two heads] \arrow[ul, dotted, "\tilde{r}^0"] \arrow[u, dotted, bend right, "r^1"'] & Y^1 \arrow[d, two heads, dashed] \arrow[u, dotted, bend right, "k^1"'] \\
\bullet \arrow[r, tail] & \bullet \arrow[r, two heads] & \bullet
\end{tikzcd}$$

of solid and dashed arrows in $\mathcal{E}$ with short exact rows and columns. By Proposition 2.12, we have $J^1 \in \mathrm{Inj}(\mathcal{E})$. The dotted arrows are now obtained as follows: The upper row and the right-hand column split since $I^0, J^1 \in \mathrm{Inj}(\mathcal{E})$. Hence, there are a left-inverse k^1 of j^1 and morphisms ι^1 and κ^0 such that $\iota^0\kappa^0 + \iota^1\kappa^1 = \mathrm{id}_{I^1}$. Using $I^0 \in \mathrm{Inj}(\mathcal{E})$ again, we lift r^0 along α^0 to obtain $\tilde{r}^0\colon X^1 \to I^0$ with $\tilde{r}^0\alpha^0 = r^0$. Set $r^1 := \iota^0\tilde{r}^0 - \iota^0\tilde{r}^0 s^1\iota^1 k^1\beta^1 + \iota^1 k^1\beta^1$. Then $r := (r^0, r^1)\colon X \to I$ is a morphism in $\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{E})$ since $\beta^1\alpha^0 = 0$ and hence $r^1\alpha^0 = \iota^0\tilde{r}^0\alpha^0 = \iota^0 r^0$. Using that $k^1\beta^1 s^1 = k^1 j^1\kappa^1 = \kappa^1$, we compute

$$\begin{aligned}
r^1 s^1 &= \iota^0\tilde{r}^0 s^1 - \iota^0\tilde{r}^0 s^1\iota^1 \circ \left(k^1\beta^1 s^1\right) + \iota^1 \circ \left(k^1\beta^1 s^1\right) = \iota^0\tilde{r}^0 s^1 \circ \left(\iota^0\kappa^0 + \iota^1\kappa^1\right) - \iota^0\tilde{r}^0 s^1\iota^1\kappa^1 + \iota^1\kappa^1 \\
&= \iota^0\tilde{r}^0 s^1\iota^0\kappa^0 + \iota^1\kappa^1 = \iota^0\tilde{r}^0\alpha^0 s^0\kappa^0 + \iota^1\kappa^1 = \iota^0 r^0 s^0\kappa^0 + \iota^1\kappa^1 = \iota^0\kappa^0 + \iota^1\kappa^1 = \mathrm{id}_{I^1},
\end{aligned}$$

as desired. □

Notation 2.22. For a Frobenius category $\mathcal{F}$ and $l \in \mathbb{N}$, we denote the stable category of $\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{F})$ by $\mathcal{M}_l := \underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$. It is a triangulated category, see Theorem 2.15.

Construction 2.23 (Projectives and injectives in $\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{E})$)**.** Let $\mathcal{E}$ be an exact category, $l \in \mathbb{N}$, and $X = (X, \alpha) \in \mathrm{Mor}_l^{\mathrm{m}}(\mathcal{E})$. If $\mathcal{E}$ has enough projectives or injectives, we construct i_X and p_X based on i_A and p_A, where $A \in \mathcal{E}$, see Convention 2.13:

(a) There is an admissible epic $p_X = (p^k)_{k=0,\dots,l}\colon P \twoheadrightarrow X$ in $\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{F})$ with $P := \bigoplus_{k=0}^{l} \mu_{l-k+1} P(X^k) \in \mathrm{Proj}(\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{F}))$ defined by

$$p^k := \begin{pmatrix} \alpha^{k-1}\cdots\alpha^0 p_{X^0} & \cdots & \alpha^{k-1} p_{X^{k-1}} & p_{X^k} \end{pmatrix} : P^k := P(X^0) \oplus \cdots \oplus P(X^k) \to X^k.$$

(b) Consider $i^0 := i_{X^0}\colon X^0 \rightarrowtail I(X^0) =: I^0$ and form a diagram

$$\begin{tikzcd}
X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, tail, "i^0"] \arrow[dr, phantom, "\square"] & X^1 \arrow[r, equal] \arrow[d, tail] & X^1 \arrow[r, tail, "\alpha^1"] \arrow[d, tail] \arrow[dr, phantom, "\square"] & X^2 \arrow[r, equal] \arrow[d, tail] & \cdots \arrow[r, equal] & X^{l-1} \arrow[r, tail, "\alpha^{l-1}"] \arrow[d, tail] \arrow[dr, phantom, "\square"] & X^l \arrow[r, equal] \arrow[d, tail] & X^l \arrow[d, tail] \\
I^0 \arrow[r, tail] \arrow[d, two heads] & W^1 \arrow[r, tail, "i^1"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & I^1 \arrow[r, tail] \arrow[d, two heads] & W^2 \arrow[r, tail, "i^2"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail, "i^{l-1}"] \arrow[dr, phantom, "\square"] & I^{l-1} \arrow[r, tail] \arrow[d, two heads] & W^l \arrow[r, tail, "i^l"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & I^l \arrow[d, two heads] \\
Y^0 \arrow[r, equal] & Y^0 \arrow[r, tail] & Y^1 \arrow[r, equal] & Y^1 \arrow[r, tail] & \cdots \arrow[r, tail] & Y^{l-1} \arrow[r, equal] & Y^{l-1} \arrow[r, tail] & Y^l
\end{tikzcd}$$

of bicartesian squares with $i^k := i_{W^k}\colon W^k \rightarrowtail I(W^k) =: I^k$ for $k \in \{1, \ldots, l\}$ and short exact columns, see Proposition 2.9. Composing horizontal monics to eliminate the columns involving $W^1, \ldots, W^l$ yields a short exact sequence $X \overset{i_X}{\rightarrowtail} I \twoheadrightarrow Y$ in $\operatorname{Mor}^{\mathrm{m}}_l(\mathcal{F})$ with $I \in \operatorname{Inj}(\operatorname{Mor}^{\mathrm{m}}_l(\mathcal{F}))$ as desired, see Theorem 2.20.

Remark 2.24. Any pushout along an admissible monic in $\operatorname{Mor}^{\mathrm{m}}_l(\mathcal{E})$ for $l \in \mathbb{N}$ yields termwise pushouts along admissible monics in an exact category $\mathcal{E}$, see Proposition 2.9.(a) and Theorem 2.20.(a). The obvious dual statement holds for pullbacks along admissible epics. In particular, for any Frobenius category $\mathcal{F}$, the (co)cone of a morphism in $\operatorname{Mor}^{\mathrm{m}}_l(\mathcal{F})$ yields termwise (co)cones in $\mathcal{F}$, see Construction 2.14 and Theorem 2.20.(b).

3. Contraction and expansion

In this section, we introduce contraction and expansion functors of stable monomorphism categories. Their respective kernels and images are the triangulated subcategories, which form various semiorthogonal decompositions, see Section 4. We give a convenient representation of their objects up to special choices of quasi-isomorphisms.

Definition 3.1. Let $\mathcal{E}$ be an exact category, $l \in \mathbb{N}$, and $s, t \in \{0, \ldots, l\}$ with $s \leq t$

(a) We define the **contraction** functor $\gamma^{[s,t]} := \gamma^{[s,t]}_l\colon \operatorname{Mor}^{\mathrm{m}}_l(\mathcal{E}) \to \operatorname{Mor}^{\mathrm{m}}_{l-t+s-1}(\mathcal{E})$ by sending an object $X = (X, \alpha) \in \operatorname{Mor}^{\mathrm{m}}_l(\mathcal{E})$ to

$$X^0 \overset{\alpha^0}{\rightarrowtail} \cdots \overset{\alpha^{s-2}}{\rightarrowtail} X^{s-1} \overset{\alpha^t \cdots \alpha^{s-1}}{\rightarrowtail} X^{t+1} \overset{\alpha^{t+1}}{\rightarrowtail} \cdots \overset{\alpha^{l-1}}{\rightarrowtail} X^l,$$

and a morphism $(f^0, \ldots, f^l)$ to $(f^0, \ldots, f^{s-1}, f^{t+1}, \ldots, f^l)$. We write $\gamma^s := \gamma^s_l := \gamma^{[s,s]}_l$,

$$\gamma^{[s,t]^c} := \gamma^{[s,t]^c}_l := \gamma^{[0,s-1]}_t \circ \gamma^{[t+1,l]}_l\colon \operatorname{Mor}^{\mathrm{m}}_l(\mathcal{F}) \to \operatorname{Mor}^{\mathrm{m}}_{t-s}(\mathcal{F}),$$

and $\gamma^{s^c} := \gamma^{s^c}_l := \gamma^{[s,s]^c}_l$.

(b) We define the **expansion** functor $\delta^{[s,t]} := \delta^{[s,t]}_l\colon \operatorname{Mor}^{\mathrm{m}}_l(\mathcal{E}) \to \operatorname{Mor}^{\mathrm{m}}_{l+t-s}(\mathcal{E})$ by sending an object $X = (X, \alpha) \in \operatorname{Mor}^{\mathrm{m}}_l(\mathcal{E})$ to

$$X^0 \overset{\alpha^0}{\rightarrowtail} \cdots \overset{\alpha^{s-1}}{\rightarrowtail} X^s = \cdots = X^s \overset{\alpha^s}{\rightarrowtail} \cdots \overset{\alpha^{l-1}}{\rightarrowtail} X^l,$$

and a morphism $(f^0, \ldots, f^l)$ to $(f^0, \ldots, f^s, \ldots, f^s, \ldots, f^l)$, where the X^s resp. f^s are placed at positions $s, \ldots, t$. We write $\delta^s := \delta^s_l := \delta^{[s,s+1]}_l$. Note that $\delta^{[s,s]}_l = \mathrm{id}_{\operatorname{Mor}^{\mathrm{m}}_l(\mathcal{E})}$.

Remark 3.2. Let $\mathcal{E}$ be an exact category, $l \in \mathbb{N}$, and $s, t \in \{0, \dots, l\}$ with $s \leq t$.

(a) We have $\gamma_{l-1}^{s} \circ \gamma_{l}^{t} = \gamma_{l-1}^{t-1} \circ \gamma_{l}^{s}$ and $\delta_{l+1}^{t} \circ \delta_{l}^{s} = \delta_{l+1}^{s} \circ \delta_{l}^{t-1}$ if $s < t$.

(b) We have $\gamma_{l}^{[s,t]} = \gamma_{l-t+s}^{s} \circ \dots \circ \gamma_{l-1}^{t-1} \circ \gamma_{l}^{t}$ and $\delta_{l}^{[s,t]} = \delta_{l-s+t-1}^{t-1} \circ \dots \circ \delta_{l+1}^{s+1} \circ \delta_{l}^{s}$. These representations are not unique, see (a).

(c) We have $\gamma_{l+t-s+1}^{[s,t]} \circ \delta_{l}^{[s,t+1]} = \mathrm{id}_{\mathrm{Mor}_{l}^{\mathrm{m}}(\mathcal{E})}$ if $t < l$, and $\gamma_{l+t-s}^{[s+1,t]} \circ \delta_{l}^{[s,t]} = \mathrm{id}_{\mathrm{Mor}_{l}^{\mathrm{m}}(\mathcal{E})}$ if $s < t$. In particular, $\delta^{[s,t]}$ is fully faithful and $\gamma^{[s,t]}$ full.

Remark 3.3. Let $\mathcal{F}$ be a Frobenius category, $l \in \mathbb{N}$, and $s, t \in \{0, \dots, l\}$ with $s \leq t$.

(a) The functors $\gamma_{l}^{[s,t]}$ and $\delta_{l}^{[s,t]}$ from Definition 3.1 are exact functors and preserve projectives, see Theorem 2.20. As such they induce triangulated functors

$$\underline{\gamma}^{[s,t]} := \underline{\gamma}_{l}^{[s,t]} \colon \underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F}) \to \underline{\mathrm{Mor}^{\mathrm{m}}}_{l-t+s-1}(\mathcal{F}) \quad \text{and} \quad \underline{\delta}^{[s,t]} := \underline{\delta}_{l}^{[s,t]} \colon \underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F}) \to \underline{\mathrm{Mor}^{\mathrm{m}}}_{l+t-s}(\mathcal{F})$$

of stable categories, see Proposition 2.18. We write $\underline{\gamma}^{s} := \underline{\gamma}_{l}^{s} := \underline{\gamma}_{l}^{[s,s]}$, $\underline{\gamma}^{[s,t]^{c}} := \underline{\gamma}_{l}^{[s,t]^{c}} := \underline{\gamma}_{t}^{[0,s-1]} \circ \underline{\gamma}_{l}^{[t+1,l]}$, $\underline{\gamma}^{s^{c}} := \underline{\gamma}_{l}^{s^{c}} := \underline{\gamma}_{l}^{[s,s]^{c}}$, and $\underline{\delta}^{s} := \underline{\delta}_{l}^{s} := \underline{\delta}^{[s,s+1]}$.

(b) Due to Remark 3.2.(c), $\underline{\gamma}_{l+t-s+1}^{[s,t]} \circ \underline{\delta}_{l}^{[s,t+1]} = \mathrm{id}_{\underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})}$ if $t < l$, and $\underline{\gamma}_{l+t-s}^{[s+1,t]} \circ \underline{\delta}_{l}^{[s,t]} = \mathrm{id}_{\underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})}$ if $s < t$. In particular, $\underline{\delta}^{[s,t]}$ is fully faithful and $\underline{\gamma}^{[s,t]}$ full.

Definition 3.4. Let $\mathcal{F}$ be a Frobenius category, $l \in \mathbb{N}$, and $s, t \in \{0, \dots, l\}$ with $s \leq t$.

(a) By $\Gamma^{[s,t]} := \Gamma_{l}^{[s,t]}$ we denote the kernel of $\underline{\gamma}_{l}^{[s,t]}$, that is, the subcategory of $\underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$ with objects X, where $X^{k} \in \mathrm{Proj}(\mathcal{F}) = \mathrm{Inj}(\mathcal{F})$ for all $k \in \{0, \dots, s-1, t+1, \dots, l\}$. We set $\Gamma^{s} := \Gamma_{l}^{s} := \Gamma_{l}^{[s,s]}$.

(b) By $\Delta^{[s,t]} := \Delta_{l}^{[s,t]}$ we denote the image of $\underline{\delta}_{l-t+s}^{[s,t]}$.

By Remark 3.3.(a), both $\Gamma_{l}^{[s,t]}$ and $\Delta_{l}^{[s,t]}$ are triangulated subcategories of $\underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$, see [Nee01, Lem. 2.1.4]

Lemma 3.5. *Let $\mathcal{F}$ be a Frobenius category and $s, t \in \{0, \dots, l\}$ with $s \leq t$. Then, $\Delta^{[s,t]}$ is the replete hull of the subcategory of $\underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$ with objects of the form*

$$\begin{tikzcd} X^{0} \arrow[r, tail] & \cdots \arrow[r, tail] & X^{s} \arrow[r, equal] & \cdots \arrow[r, equal] & X^{t} \arrow[r, tail] & \cdots \arrow[r, tail] & X^{l}. \end{tikzcd}$$

The restriction of $\underline{\gamma}^{[s,t-1]}$ or, equally, $\underline{\gamma}^{[s+1,t]}$ to $\Delta^{[s,t]}$ is quasi-inverse to the triangle equivalence $\underline{\mathrm{Mor}^{\mathrm{m}}}_{l+s-t}(\mathcal{F}) \xrightarrow{\simeq} \Delta^{[s,t]}$ given by $\underline{\delta}^{[s,t]}$.

Proof. The first claim holds by definition, the second due to Remarks 3.3.(b) and 3.6. □

Remark 3.6. Let $\eta \colon F \to G$ be a natural transformation of functors $F, G \colon \mathcal{A} \to \mathcal{B}$. If $H \colon \mathcal{B} \to C$ is another functor, then the *whiskering* $H\eta := (H(\eta_{X}))_{X \in \mathcal{A}}$ of η on the right by H is a natural transformation $HF \to HG$. It is an isomorphism if η is so. In particular, if F is an equivalence with quasi-inverse F^{-1} and $F' : \mathcal{B} \to \mathcal{A}$ a functor with $F'F = \mathrm{id}_{\mathcal{A}}$, then $F^{-1} = F'FF^{-1} \simeq F'$. So, F' is already a quasi-inverse of F.

Lemma 3.7. *Let $\mathcal{E}$ be an exact category, $l \in \mathbb{N}$, and $s, t \in \{0, \dots, l\}$ with $s \leq t$. Given $X = (X, \alpha) \in \mathrm{Mor}_{l}^{\mathrm{m}}(\mathcal{E})$ with $X^{k} \in \mathrm{Inj}(\mathcal{E})$ for all $k \in \{0, \dots, s-1, t+1, \dots, l\}$, there is* *(a) a split short exact sequence*

$$\begin{tikzcd}
I \arrow[d, tail] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail, "\alpha^{s-2}"] \arrow[dr, phantom, "\square"] & X^{s-1} \arrow[r, equal] \arrow[d, equal] & X^{s-1} \arrow[r, equal] \arrow[d, tail, "\alpha^{s-1}"] & \cdots \arrow[r, equal] & X^{s-1} \arrow[r, equal] \arrow[d, tail, "\alpha^{t-1}\cdots\alpha^{s-1}"'] & I^{t+1} \arrow[r, tail] \arrow[d, tail, "\alpha^{t}\cdots\alpha^{s-1}"'] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & I^l \arrow[d, tail] \\
X \arrow[d, two heads, "p"] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, two heads, "p^0"] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^{s-1}"] \arrow[d, two heads, "p^{s-1}"'] \arrow[dr, phantom, "\square"] & X^s \arrow[r, tail, "\alpha^s"] \arrow[d, two heads, "p^s"] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail, "\alpha^{t-1}"] \arrow[dr, phantom, "\square"] & X^t \arrow[r, tail, "\alpha^t"] \arrow[d, two heads, "p^t"] \arrow[dr, phantom, "\square"] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d, two heads, "p^{t+1}"] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d, two heads, "p^l"] \\
\tilde{X} & 0 \arrow[r, equal] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] & \tilde{X}^s \arrow[r, tail] & \cdots \arrow[r, tail] & \tilde{X}^t \arrow[r, tail] & \tilde{X}^{t+1} \arrow[r, equal] & \cdots \arrow[r, equal] & \tilde{X}^{t+1}
\end{tikzcd}$$

(b) with reverse split short exact sequence

$$\begin{tikzcd}
\tilde{X} \arrow[d, tail, "i"] & 0 \arrow[r, equal] \arrow[d, tail, "i^0"] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] \arrow[d, tail, "i^{s-1}"] \arrow[dr, phantom, "\square"] & \tilde{X}^s \arrow[r, tail] \arrow[d, tail, "i^s"] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & \tilde{X}^t \arrow[r, tail] \arrow[d, tail, "i^t"] \arrow[dr, phantom, "\square"] & \tilde{X}^{t+1} \arrow[r, equal] \arrow[d, tail, "i^{t+1}"] & \cdots \arrow[r, equal] & \tilde{X}^{t+1} \arrow[d, tail, "i^l"] \\
X \arrow[d, two heads] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail, "\alpha^{s-2}"] \arrow[dr, phantom, "\square"] & X^{s-1} \arrow[r, tail, "\alpha^{s-1}"] \arrow[d, equal] & X^s \arrow[r, tail, "\alpha^s"] \arrow[d, two heads] & \cdots \arrow[r, tail, "\alpha^{t-1}"] & X^t \arrow[r, tail, "\alpha^t"] \arrow[d, two heads] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail, "\alpha^{l-1}"] \arrow[dr, phantom, "\square"] & X^l \arrow[d, two heads] \\
I & X^0 \arrow[r, tail, "\alpha^0"] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, equal] & X^{s-1} \arrow[r, equal] & \cdots \arrow[r, equal] & X^{s-1} \arrow[r, equal] & I^{t+1} \arrow[r, tail] & \cdots \arrow[r, tail] & I^l
\end{tikzcd}$$

in $\operatorname{Mor}^{\mathrm{m}}_l(\mathcal{E})$*, where* $\tilde{X}^{t+1} \in \operatorname{Inj}(\mathcal{E})$*,* $I \in \operatorname{Inj}(\operatorname{Mor}^{\mathrm{m}}_l(\mathcal{E}))$*, and* $X^{-1} := 0$*. The morphism* i^{t+1} *can be chosen as an arbitrary right-inverse of* p^{t+1}*.*
In particular, if $\mathcal{E}$ *is Frobenius, then* p *and* i *are quasi-isomorphisms and* $\Gamma_l^{[s,t]}$ *is the replete hull of the subcategory of* $\underline{\operatorname{Mor}^{\mathrm{m}}}_l(\mathcal{E})$ *with objects of the form*

$$\begin{tikzcd}
0 \arrow[r, equal] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] & X^s \arrow[r, tail] & \cdots \arrow[r, tail] & X^t \arrow[r, tail] & X^{t+1} \arrow[r, equal] & \cdots \arrow[r, equal] & X^{t+1},
\end{tikzcd}$$

where $X^{t+1} \in \operatorname{Proj}(\mathcal{E}) = \operatorname{Inj}(\mathcal{E})$*.*

Proof.

(a) Due to Proposition 2.9.(a), forming successive pushouts yields a short exact sequence

$$\begin{tikzcd}
J \arrow[d, tail] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, equal] \arrow[d, equal] & X^{s-1} \arrow[r, equal] \arrow[d, tail, "\alpha^{s-1}"] & \cdots \arrow[r, equal] & X^{s-1} \arrow[r, equal] \arrow[d, tail, "\alpha^{t-1}\cdots\alpha^{s-1}"] & X^{s-1} \arrow[r, equal] \arrow[d, tail, "\alpha^{t}\cdots\alpha^{s-1}"] & \cdots \arrow[r, equal] & X^{s-1} \arrow[d, tail, "\alpha^{l-1}\cdots\alpha^{s-1}"] \\
X \arrow[d, two heads] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, two heads] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^{s-1}"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & X^s \arrow[r, tail, "\alpha^s"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail, "\alpha^{t-1}"] \arrow[dr, phantom, "\square"] & X^t \arrow[r, tail, "\alpha^t"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail, "\alpha^{l-1}"] \arrow[dr, phantom, "\square"] & X^l \arrow[d, two heads] \\
Y & 0 \arrow[r, equal] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] & \tilde{X}^s \arrow[r, tail] & \cdots \arrow[r, tail] & \tilde{X}^t \arrow[r, tail] & \tilde{X}^{t+1} \arrow[r, tail] & \cdots \arrow[r, tail] & \tilde{X}^l
\end{tikzcd} \tag{3.1}$$

in $\operatorname{Mor}^{\mathrm{m}}_l(\mathcal{E})$. If $s = 0$, use $\mathrm{id}_{X^0} \colon X^0 \to X^0 =: \tilde{X}^0$ for the leftmost pushout. By Theorem 2.20.(b), $J \in \operatorname{Inj}(\operatorname{Mor}^{\mathrm{m}}_l(\mathcal{E}))$ and the sequence splits. In particular, $X \twoheadrightarrow Y$ is a quasi-isomorphism if $\mathcal{E}$ is Frobenius, see Lemma 2.17. If $t = l$, this proves the claim with $\tilde{X} = Y$ and $I = J$.
Otherwise, for each $k \in \{t+1, \dots, l\}$, the short exact sequence $X^{s-1} \rightarrowtail X^k \twoheadrightarrow \tilde{X}^k$ yields $\tilde{X}^k \in \operatorname{Inj}(\mathcal{E})$

such that the admissible monic $\tilde{X}^{t+1} \rightarrowtail \tilde{X}^k$ splits, which gives rise to a biproduct $\tilde{X}^k \cong \tilde{X}^{t+1} \oplus \tilde{J}^k$ with $\tilde{J}^k \in \mathrm{Inj}(\mathcal{E})$, see Proposition 2.12. Lemma 3.8 applied to $\gamma^{[0,t]}(Y)$ then yields a short exact sequence

$$\begin{tikzcd}
\tilde{J} \arrow[d, tail] & 0 \arrow[r, equal] \arrow[d, equal] & \cdots \arrow[r, equal] & 0 \arrow[r, equal] \arrow[d, equal] & 0 \arrow[r, equal] \arrow[d, tail] & \cdots \arrow[r, equal] & 0 \arrow[r, equal] \arrow[d, tail] & 0 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \tilde{J}^{t+2} \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & \tilde{J}^l \arrow[d, tail] \\
Y \arrow[d, two heads] & 0 \arrow[r, equal] \arrow[d, equal] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] \arrow[d, equal] & \tilde{X}^s \arrow[r, tail] \arrow[d, equal] & \cdots \arrow[r, tail] & \tilde{X}^t \arrow[r, tail] \arrow[d, equal] & \tilde{X}^{t+1} \arrow[r, tail] \arrow[d, equal] & \tilde{X}^{t+2} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & \tilde{X}^l \arrow[d, two heads] \\
\tilde{X} & 0 \arrow[r, equal] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] & \tilde{X}^s \arrow[r, tail] & \cdots \arrow[r, tail] & \tilde{X}^t \arrow[r, tail] & \tilde{X}^{t+1} \arrow[r, equal] & \tilde{X}^{t+1} \arrow[r, equal] & \cdots \arrow[r, equal] & \tilde{X}^{t+1}
\end{tikzcd}$$

in $\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{E})$ with $\tilde{J} \in \mathrm{Inj}(\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{E}))$, see Theorem 2.20.(b). As above, $Y \twoheadrightarrow \tilde{X}$ splits and is a quasi-isomorphism if $\mathcal{E}$ is Frobenius. Due to Lemma 3.9, $I := J \oplus \tilde{J}$ is the kernel of the composition $p\colon X \twoheadrightarrow Y \twoheadrightarrow \tilde{X}$.

(b) A right-inverse i^{t+1} of p^{t+1} corresponds to a left-inverse $r^{t+1}\colon X^{t+1} \twoheadrightarrow I^{t+1}$ of $\alpha^t \cdots \alpha^{s-1}$. Set $r^k := r^{t+1}\alpha^t \cdots \alpha^k$ for $k \in \{s, \dots, t\}$ and $r^k := \mathrm{id}_{X^k}$ for $k \in \{0, \dots, s-1\}$. Apply $\gamma^{[0,t]}$ followed by Lemma 2.21 to define r^k for $k \in \{t+2, \dots, l\}$ starting from r^{t+1}. Then $r := \left(r^0, \dots, r^l\right) : X \to I$ is a left-inverse of $I \rightarrowtail X$, which corresponds to the desired right-inverse of p. The claimed bicartesian squares are due to Proposition 2.9. □

Lemma 3.8. *Let $\mathcal{E}$ be an exact category and $l \in \mathbb{N}$. Then any $X = (X, \alpha) \in \mathrm{Mor}_l^{\mathrm{sm}}(\mathcal{E})$ fits into a termwise split short exact sequence*

$$\begin{tikzcd}
0 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \tilde{X}^1 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \tilde{X}^2 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & \tilde{X}^l \arrow[d, tail] \\
X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] & X^1 \arrow[r, tail, "\alpha^1"] \arrow[d, two heads, "\tilde{\beta}^1"] & X^2 \arrow[r, tail, "\alpha^2"] \arrow[d, two heads, "\tilde{\beta}^2"] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d, two heads, "\tilde{\beta}^l"] \\
X^0 \arrow[r, equal] & X^0 \arrow[r, equal] & X^0 \arrow[r, equal] & \cdots \arrow[r, equal] & X^0
\end{tikzcd}$$

in $\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{E})$, where $\tilde{X}^k$ is the cokernel of $\tilde{\alpha}^{k-1} := \alpha^{k-1} \cdots \alpha^0 \colon X^0 \rightarrowtail X^k$ for $k \in \{1, \dots, k\}$.

Proof. By assumption, for any $k \in \{1, \dots, l\}$, there is $\beta^k\colon X^k \to X^{k-1}$ such that $\beta^k \alpha^{k-1} = \mathrm{id}_{X^{k-1}}$. In particular, $\tilde{\beta}^k := \beta^1 \cdots \beta^k\colon\ : X^k \to X^0$ satisfies $\tilde{\beta}^k \tilde{\alpha}^{k-1} = \mathrm{id}_{X^0}$ and $\tilde{\beta}^k \alpha^{k-1} = \tilde{\beta}^{k-1}$, which establishes the vertical split short exact sequences and the lower half of the diagram. Lemma 2.10 applied to

$$\begin{tikzcd}
\tilde{X}^k \arrow[r, tail, dashed] \arrow[d, tail] & \tilde{X}^{k+1} \arrow[r, two heads, dashed] \arrow[d, tail] & \bullet \arrow[d, equal] \\
X^k \arrow[r, tail, "\alpha^k"] \arrow[d, two heads, "\tilde{\beta}^k"] & X^{k+1} \arrow[r, two heads] \arrow[d, two heads, "\tilde{\beta}^k"] & \bullet \arrow[d, two heads] \\
X^0 \arrow[r, equal] & X^0 \arrow[r, two heads] & 0
\end{tikzcd}$$

for $k \in \{1, \dots, l-1\}$ then completes the diagram. The bicartesian squares are due to Proposition 2.9.(b). $\square$

Lemma 3.9. *In an exact category, any split short exact sequence $A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} C$ and short exact sequence $A' \overset{i'}{\rightarrowtail} C \overset{p'}{\twoheadrightarrow} D$ give rise to a short exact sequence*

$$A \oplus A' \xrightarrow{\begin{pmatrix} i & ji' \end{pmatrix}} B \xrightarrow{p'p} D,$$

where $j \colon C \to B$ is any right-inverse of p.

Proof. Given a kernel $k \colon E \rightarrowtail B$ of $p'p$, Lemma 2.10 yields a commutative diagram

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, tail, dashed, "t"] & A \arrow[r, two heads] \arrow[d, tail, "i"'] & 0 \arrow[d, tail] \\
E \arrow[r, tail, "k"] \arrow[d, dashed, two heads] & B \arrow[r, two heads, "p'p"] \arrow[d, two heads, "p"'] \arrow[u, bend right, dotted, "q"'] & D \arrow[d, equal] \\
A' \arrow[r, tail, "i'"] & C \arrow[r, two heads, "p'"] \arrow[u, bend right, dotted, "j"'] & D
\end{tikzcd}$$

of solid and dashed arrows with short exact rows and columns. With q the left-inverse of i corresponding to j, then $qkt = qi = \mathrm{id}_A$ and the left column splits. With respect to this splitting, k corresponds to $\begin{pmatrix} i & ji' \end{pmatrix}$ and the claim follows. $\square$

4. Semiorthogonal decompositions

In this section, we establish various semiorthogonal decompositions of stable monomorphism categories using the subcategories introduced in Section 3. These decompositions form polygons of recollements.

Proposition 4.1. *Let $\mathcal{F}$ be a Frobenius category, $l \in \mathbb{N}$ and $s \in \{0, \dots, l-1\}$. Then $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ admits the semiorthogonal decomposition $\left(\Gamma^{[s+1,l]}, \Gamma^{[0,s]}\right)$ as follows:*

(a) Any $X = (X, \alpha) \in \mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$ fits into a distinguished triangle in $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ represented by the commutative diagram in $\mathcal{F}$

$$\begin{tikzcd}
0 \arrow[r, equal] \arrow[d] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] \arrow[d] & X^{s+1} \arrow[r, tail, "\alpha^{s+1}"] \arrow[d, equal] & X^{s+2} \arrow[r, tail, "\alpha^{s+2}"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d, equal] \\
X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{s-1}"] & X^s \arrow[r, tail, "\alpha^s"] \arrow[d, equal] & X^{s+1} \arrow[r, tail, "\alpha^{s+1}"] \arrow[d, tail, "i_{X^{s+1}}"] & X^{s+2} \arrow[r, tail, "\alpha^{s+2}"] \arrow[d] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d] \\
X^0 \arrow[r, tail, "\alpha^0"] & \cdots \arrow[r, tail, "\alpha^{s-1}"] & X^s \arrow[r, tail] & I(X^{s+1}) \arrow[r, phantom, "="] & I(X^{s+1}) \arrow[r, equal] & \cdots \arrow[r, equal] & I(X^{s+1}).
\end{tikzcd}$$

(b) The inclusion $\Gamma^{[s+1,l]} \hookrightarrow \underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ has a right adjoint which sends any $X \in \mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$ to the object in the upper row of the diagram in (a).

(c) *The inclusion* $\Gamma^{[0,s]} \hookrightarrow \underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ *has a left adjoint which sends any* $X \in \mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$ *to the object in the lower row of the diagram in (a).*

Proof. For condition (a) in Definition 2.1, consider a morphism $f\colon X \to Y$ in $\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$ with $X \in \Gamma^{[s+1,l]}$ and $Y \in \Gamma^{[0,s]}$. Due to Lemma 3.7, we may assume that $X^k = 0$ for $k \in \{0, \dots, s\}$ and $Y^k \in \mathrm{Proj}(\mathcal{F})$ for $k \in \{s+1, \dots, l\}$. Then f factors as follows:

$$\begin{tikzcd}
0 \arrow[r, equal] \arrow[d, equal] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] \arrow[d, equal] & X^{s+1} \arrow[r, tail] \arrow[d, "f^{s+1}"] & \cdots \arrow[r, tail] & X^l \arrow[d, "f^l"] \\
0 \arrow[r, equal] \arrow[d, "f^0"] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] \arrow[d, "f^s"] & Y^{s+1} \arrow[r, tail] \arrow[d, equal] & \cdots \arrow[r, tail] & Y^l \arrow[d, equal] \\
Y^0 \arrow[r, tail] & \cdots \arrow[r, tail] & Y^s \arrow[r, tail] & Y^{s+1} \arrow[r, tail] & \cdots \arrow[r, tail] & Y^l
\end{tikzcd}$$

The middle object is projective, see Theorem 2.20.(b), and hence $f = 0$ in $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$.
For condition (b) in Definition 2.1, we prove that for any $X \in \mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$ the morphism

$$\begin{tikzcd}
\tilde{X} \arrow[d, "f"] & 0 \arrow[r, equal] \arrow[d] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] \arrow[d] & X^{s+1} \arrow[r, tail] \arrow[d, equal] & \cdots \arrow[r, tail] & X^l \arrow[d, equal] \\
X & X^0 \arrow[r, tail] & \cdots \arrow[r, tail] & X^s \arrow[r, tail] & X^{s+1} \arrow[r, tail] & \cdots \arrow[r, tail] & X^l
\end{tikzcd}$$

in $\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$ with $\tilde{X} \in \Gamma^{[s+1,l]}$ has its cone $C := C(f)$ in $\Gamma^{[0,s]}$. To this end, we consider the pushout of f along the admissible monic $i := i_{\tilde{X}}\colon \tilde{X} \rightarrowtail I(\tilde{X}) =: I$, where $I^k = 0$ for all $k \in \{0, \dots, s\}$ and $i^{s+1} = i_{X^{s+1}}\colon X^{s+1} \to I(X^{s+1}) = I^{s+1}$, see Convention 2.13, Construction 2.23.(b), and Remark 2.24:

$$\begin{tikzcd}
 & I \arrow[dd] & & 0 \arrow[rr, equal] \arrow[dl, equal] \arrow[dd] & & \cdots \arrow[rr, equal] & & 0 \arrow[rr, tail] \arrow[dl, equal] \arrow[dd] & & I^{s+1} \arrow[rr, tail] \arrow[dd, equal] & & I^{s+2} \arrow[rr, tail] \arrow[dd, equal] & & \cdots \arrow[rr, tail] & & I^l \arrow[dd, equal] \\
\tilde{X} \arrow[ur, tail, "i"] \arrow[dd, "f"'] & & 0 \arrow[rr, equal] \arrow[dd] & & \cdots \arrow[rr, equal] & & 0 \arrow[rr, tail] \arrow[dd] & & X^{s+1} \arrow[rr, tail] \arrow[ur, tail] \arrow[dd, equal] & & X^{s+2} \arrow[rr, tail] \arrow[ur, tail] \arrow[dd, equal] & & \cdots \arrow[rr, tail] & & X^l \arrow[ur, tail] \arrow[dd, equal] & \\
 & C & & X^0 \arrow[rr, tail] & & \cdots \arrow[rr, tail] & & X^s \arrow[rr, tail] & & I^{s+1} \arrow[rr, tail] & & I^{s+2} \arrow[rr, tail] & & \cdots \arrow[rr, tail] & & I^l \\
X \arrow[ur, tail] & & X^0 \arrow[rr, tail] \arrow[ur, equal] & & \cdots \arrow[rr, tail] & & X^s \arrow[rr, tail] \arrow[ur, equal] & & X^{s+1} \arrow[rr, tail] \arrow[ur, tail, "i^{s+1}"'] & & X^{s+2} \arrow[rr, tail] \arrow[ur, tail] & & \cdots \arrow[rr, tail] & & X^l \arrow[ur, tail] &
\end{tikzcd}$$

We postcompose $X \to C$ with the quasi-isomorphism $C \to \tilde{C}$ from Lemma 3.7.(a) to obtain a distinguished triangle

$$\begin{tikzcd}
\tilde{X} \arrow[d, "f"] & 0 \arrow[r, equal] \arrow[d] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] \arrow[d] & X^{s+1} \arrow[r, tail] \arrow[d, equal] & X^{s+2} \arrow[r, tail] \arrow[d, equal] & \cdots \arrow[r, tail] & X^l \arrow[d, equal] \\
X \arrow[d] & X^0 \arrow[r, tail] \arrow[d, equal] & \cdots \arrow[r, tail] & X^s \arrow[r, tail] \arrow[d, equal] & X^{s+1} \arrow[r, tail] \arrow[d, tail, "i_{X^{s+1}}"] & X^{s+2} \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & X^l \arrow[d] \\
\tilde{C} & X^0 \arrow[r, tail] & \cdots \arrow[r, tail] & X^s \arrow[r, tail] & I(X^{s+1}) \arrow[r, phantom, "="] & I(X^{s+1}) \arrow[r, equal] & \cdots \arrow[r, equal] & I(X^{s+1})
\end{tikzcd} \tag{4.1}$$

in $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ with $\tilde{X} \in \Gamma^{[s+1,l]}$ and $\tilde{C} \in \Gamma^{[0,s]}$. The claims on adjoints is due to Proposition 2.3. □

Corollary 4.2. *Let $\mathcal{F}$ be a Frobenius category, $l \in \mathbb{N}$, and $s \in \{0, \dots, l\}$. Given $f\colon X \to Y \in \mathrm{Mor}^{\mathrm{m}}_s(\mathcal{F})$ and two admissible monics $i\colon X^s \rightarrowtail I$ and $j\colon Y^s \rightarrowtail J$ in $\mathcal{F}$ with $I, J \in \mathrm{Proj}(\mathcal{F})$, there is a unique morphism of the form*

$$\begin{tikzcd}
\tilde{X} \arrow[d, "\tilde{f}"] & X^0 \arrow[r, tail] \arrow[d, "f^0"] & \cdots \arrow[r, tail] & X^s \arrow[r, tail, "i"] \arrow[d, "f^s"] & I \arrow[r, equal] \arrow[d] & \cdots \arrow[r, equal] & I \arrow[d] \\
\tilde{Y} & Y^0 \arrow[r, tail] & \cdots \arrow[r, tail] & Y^s \arrow[r, tail, "j"] & J \arrow[r, equal] & \cdots \arrow[r, equal] & J.
\end{tikzcd}$$

in $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$. In particular, $\tilde{f}$ is an isomorphism if $f = \mathrm{id}_X$.

Proof. The objects $\tilde{X}, \tilde{Y} \in \Gamma^{[0,s-1]}$ appear in the distinguished triangles of Proposition 4.1.(a) applied to $\delta^{[s,l]}_s(X)$ with $i_{X^s} = i$ and $\delta^{[s,l]}_s(Y)$ with $i_{Y^s} = j$. Remark 2.2 yields the claim. □

Combining the particular claims of Lemma 3.7 and Corollary 4.2 yields

Corollary 4.3. *Let $\mathcal{F}$ be a Frobenius category, $l \in \mathbb{N}$, and $s, t \in \{0, \dots, l\}$ with $s \le t$. Then $\Gamma^{[s,t]} \subseteq \underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ is the replete hull of the subcategory of $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ with objects of the form*

$$0 = \cdots = 0 \rightarrowtail X^s \rightarrowtail \cdots \rightarrowtail X^t \overset{i_{X^t}}{\rightarrowtail} I(X^t) = \cdots = I(X^t).$$

□

Proposition 4.4. *Let $\mathcal{F}$ be a Frobenius category and $l \in \mathbb{N}$.*

(a) For $s, t \in \{0, \dots, l-1\}$ with $s \le t$, $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ admits the semiorthogonal decomposition $\left(\Gamma^{[s,t]}, \Delta^{[s,t+1]}\right)$ as follows:

(i) Any $X = (X, \alpha) \in \mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$ fits into a distinguished triangle in $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ represented by the commutative diagram in $\mathcal{F}$

$$\begin{tikzcd}
0 \arrow[r, equal] \arrow[d] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] \arrow[d] & Y^s \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & Y^t \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & P(X^{t+1}) \arrow[r, equal] \arrow[d, two heads, "p_{X^{t+1}}"] & \cdots \arrow[r, equal] & P(X^{t+1}) \arrow[d] \\
X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^{s-1}"] \arrow[d, equal] & X^s \arrow[r, tail, "\alpha^s"] \arrow[d, tail, "\alpha^t \cdots \alpha^s"] & \cdots \arrow[r, tail, "\alpha^{t-1}"] & X^t \arrow[r, tail, "\alpha^t"] \arrow[d, tail, "\alpha^t"] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{l-2}"] & X^l \arrow[d, equal] \\
X^0 \arrow[r, tail, "\alpha^0"] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^t \cdots \alpha^{s-1}"] & X^{t+1} \arrow[r, equal] & \cdots \arrow[r, equal] & X^{t+1} \arrow[r, equal] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l.
\end{tikzcd}$$

(ii) The inclusion $\Gamma^{[s,t]} \hookrightarrow \underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ has a right adjoint which sends any $X \in \mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$ to the object in the upper row of the diagram in (i).

(iii) For $s < t$, the inclusion $\Delta^{[s,t]} \hookrightarrow \underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ has a left adjoint which sends any $X \in \mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$ to the object $\delta^{[s,t]}\gamma^{[s,t-1]}(X)$ in the lower row of the diagram in (i).

(b) For $s, t \in \{1, \dots, l\}$ with $s \le t$, $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ admits the semiorthogonal decomposition $\left(\Delta^{[s-1,t]}, \Gamma^{[s,t]}\right)$ as follows:

(i) Any $X = (X, \alpha) \in \mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$ fits into a distinguished triangle in $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ represented by the commutative diagram in $\mathcal{F}$

$$\begin{tikzcd}
X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, equal] \arrow[d, equal] & X^{s-1} \arrow[r, equal] \arrow[d, tail, "\alpha^{s-1}"] & \cdots \arrow[r, equal] & X^{s-1} \arrow[r, tail, "\alpha^t\cdots\alpha^{s-1}"] \arrow[d, tail, "\alpha^{t-1}\cdots\alpha^{s-1}"] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d, equal] \\
X^0 \arrow[r, tail, "\alpha^0"] \arrow[d] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^{s-1}"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & X^s \arrow[r, tail, "\alpha^s"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail, "\alpha^{t-1}"] \arrow[dr, phantom, "\square"] & X^t \arrow[r, tail, "\alpha^t"] \arrow[d, two heads] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d] \\
0 \arrow[r, equal] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] & Y^s \arrow[r, tail] & \cdots \arrow[r, tail] & Y^t \arrow[r, tail, "i_{Y^t}"] & I(Y^t) \arrow[r, equal] & \cdots \arrow[r, equal] & I(Y^t).
\end{tikzcd}$$

(ii) *For $s < t$, the inclusion $\Delta^{[s,t]} \hookrightarrow \underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ has a right adjoint which sends any $X \in \mathrm{Mor}_l^{\mathrm{m}}(\mathcal{F})$ to the object $\delta^{[s,t]}\gamma^{[s+1,t]}(X)$ in the upper row of the diagram in (i).*

(iii) *The inclusion $\Gamma^{[s,t]} \hookrightarrow \underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ has a left adjoint which sends any $X \in \mathrm{Mor}_l^{\mathrm{m}}(\mathcal{F})$ to the object in the lower row of the diagram in (i).*

Proof.

(a) For condition (a) in Definition 2.1, consider a morphism $f\colon (X,\alpha) \to (Y,\beta)$ in $\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{F})$ with $X \in \Gamma^{[s,t]}$ and $Y \in \Delta^{[s,t+1]}$. It factors as

$$\begin{tikzcd}
X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^{s-1}"] \arrow[d, equal] & X^s \arrow[r, tail, "\alpha^s"] \arrow[d, tail, "\alpha^t\cdots\alpha^s"] & \cdots \arrow[r, tail, "\alpha^{t-1}"] & X^t \arrow[r, tail, "\alpha^t"] \arrow[d, tail, "\alpha^t"] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d, equal] & X^{t+2} \arrow[r, tail, "\alpha^{t+2}"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d, equal] \\
X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, "f^0"] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^{s-1}"] \arrow[d, "f^{s-1}"] & X^{t+1} \arrow[r, equal] \arrow[d, "f^{t+1}"] & \cdots \arrow[r, equal] & X^{t+1} \arrow[r, equal] \arrow[d, "f^{t+1}"] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d, "f^{t+1}"] & X^{t+2} \arrow[r, tail, "\alpha^{t+2}"] \arrow[d, "f^{t+2}"] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d, "f^l"] \\
Y^0 \arrow[r, tail, "\beta^0"] & \cdots \arrow[r, tail, "\beta^{s-2}"] & Y^{s-1} \arrow[r, tail, "\beta^{s-1}"] & Y^s \arrow[r, equal] & \cdots \arrow[r, equal] & Y^t \arrow[r, equal] & Y^{t+1} \arrow[r, tail, "\beta^{t+1}"] & Y^{t+2} \arrow[r, tail, "\beta^{t+2}"] & \cdots \arrow[r, tail, "\beta^{l-1}"] & Y^l,
\end{tikzcd}$$

since $\beta^k = \mathrm{id}_{Y^k}$ and hence $f^k = f^{t+1}\alpha^t \cdots \alpha^k$ for $k \in \{s, \ldots, t\}$. The object in the middle row is projective since $X^k \in \mathrm{Proj}(\mathcal{F})$ for $k \in \{0, \ldots, s-1, t+1, \ldots, l\}$, see Theorem 2.20.(b), and hence $f = 0$ in $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$.

For condition (b) in Definition 2.1, let $X = (X, \alpha) \in \mathrm{Mor}_l^{\mathrm{m}}(\mathcal{F})$ be arbitrary, and set $\tilde{X} := \delta^{[s,t+1]}\gamma^{[s,t]}(X) \in \Delta^{[s,t+1]}$. We prove that the morphism

$$\begin{tikzcd}
X \arrow[d, "f"'] & & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^{s-1}"] \arrow[d, equal] & X^s \arrow[r, tail, "\alpha^s"] \arrow[d, tail, "\alpha^t\cdots\alpha^s"] & \cdots \arrow[r, tail, "\alpha^{t-1}"] & X^t \arrow[r, tail, "\alpha^t"] \arrow[d, tail, "\alpha^t"] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d, equal] \\
\tilde{X} & & X^0 \arrow[r, tail, "\alpha^0"] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^t\cdots\alpha^{s-1}"] & X^{t+1} \arrow[r, equal] & \cdots \arrow[r, equal] & X^{t+1} \arrow[r, equal] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] & \cdots \arrow[r, tail, "\alpha^l"] & X^l
\end{tikzcd}$$

in $\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{F})$ has its cocone $C^* := C^*(f)$ in $\Gamma^{[s,t]}$. To this end, we consider the pullback of f along the admissible epic $p := \delta^{[s,t+1]}(p_{\gamma^{[s,t]}(X)})\colon P \twoheadrightarrow \tilde{X}$ with $P \in \mathrm{Mor}_l^{\mathrm{m}}(\mathrm{Proj}(\mathcal{F}))$ and

$$p^k = \begin{pmatrix} \alpha^t \cdots \alpha^{s-1} p^{s-1} & p_{X^{t+1}} \end{pmatrix} : P^k = P^{s-1} \oplus P(X^{t+1}) \twoheadrightarrow X^{t+1} \tag{4.2}$$

for all $k \in \{s, \ldots, t+1\}$, see Construction 2.23.(a). Due to Remark 2.24, it displays as a termwise pullback:

$$\begin{tikzcd}[column sep=small]
& X \arrow[dd, "f"] & & X^0 \arrow[rr, tail] \arrow[dd, equal] & & \cdots \arrow[rr, tail] & & X^{s-1} \arrow[rr, tail] \arrow[dd, equal] & & X^s \arrow[rr, tail] \arrow[dd, tail] & & \cdots \arrow[rr, tail] & & X^t \arrow[rr, tail] \arrow[dd, tail] & & X^{t+1} \arrow[rr, tail] \arrow[dd, equal] & & \cdots \arrow[rr, tail] & & X^l \arrow[dd, equal] \\
C^* \arrow[ur, two heads] \arrow[dd] & & P^0 \arrow[ur, two heads] \arrow[rr, tail, crossing over] \arrow[dd, equal] & & \cdots \arrow[rr, tail] & & P^{s-1} \arrow[ur, two heads] \arrow[rr, tail, crossing over] \arrow[dd, equal] & & C^s \arrow[ur, two heads] \arrow[rr, tail, crossing over] \arrow[dd, tail] & & \cdots \arrow[rr, tail] & & C^t \arrow[ur, two heads] \arrow[rr, tail, crossing over] \arrow[dd, tail] & & P^{t+1} \arrow[ur, two heads] \arrow[rr, tail, crossing over] \arrow[dd, equal] & & \cdots \arrow[rr, tail] & & P^l \arrow[ur, two heads] \arrow[dd, equal] & \\
& \tilde{X} & & X^0 \arrow[rr, tail] & & \cdots \arrow[rr, tail, crossing over] & & X^{s-1} \arrow[rr, tail, crossing over] & & X^{t+1} \arrow[rr, equal] & & \cdots \arrow[rr, equal, crossing over] & & X^{t+1} \arrow[rr, equal, crossing over] & & X^{t+1} \arrow[rr, tail] & & \cdots \arrow[rr, tail, crossing over] & & X^l \\
P \arrow[ur, two heads, "p"'] & & P^0 \arrow[ur, two heads] \arrow[rr, tail] & & \cdots \arrow[rr, tail] & & P^{s-1} \arrow[ur, two heads] \arrow[rr, tail] & & P^s \arrow[ur, two heads] \arrow[rr, equal] & & \cdots \arrow[rr, equal] & & P^t \arrow[ur, two heads] \arrow[rr, equal] & & P^{t+1} \arrow[ur, two heads] \arrow[rr, tail] & & \cdots \arrow[rr, tail] & & P^l \arrow[ur, two heads] &
\end{tikzcd}$$

In particular, for each $k \in \{s, \ldots, t\}$, the square

$$\begin{tikzcd}
C^k \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & P^k = P^{t+1} \arrow[d, two heads] \\
X^k \arrow[r, tail, "\alpha^t\cdots\alpha^k"] & X^{t+1}
\end{tikzcd}$$

is bicartesian and, for $k < t$, so is the left square in the diagram

$$\begin{tikzcd}
C^k \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & C^{k+1} \arrow[r, tail] \arrow[d, two heads] & P^{t+1} \arrow[d, two heads] \\
X^k \arrow[r, tail, "\alpha^k"] & X^{k+1} \arrow[r, tail, "\alpha^t\cdots\alpha^{k+1}"] & X^{t+1}
\end{tikzcd}$$

due to Lemma 4.6.(b). Using the right-inverse of $P^{t+1} \twoheadrightarrow P(X^{t+1})$ given by the biproduct in (4.2), Lemma 3.7.(b) yields a quasi-isomorphism $\tilde{C}^* \to C^*$. By composition we obtain a quasi-isomorphic cocone of f:

$$\begin{tikzcd}
\tilde{C}^* \arrow[d] & 0 \arrow[r, equal] \arrow[d, tail] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{C}^s \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & \tilde{C}^t \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & P(X^{t+1}) \arrow[r, equal] \arrow[d, tail] & \cdots \arrow[r, equal] & P(X^{t+1}) \arrow[d, tail] \\
C^* \arrow[d] & P^0 \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & P^{s-1} \arrow[r, tail] \arrow[d, two heads] & C^s \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & C^t \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & P^{t+1} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & P^l \arrow[d, two heads] \\
X & X^0 \arrow[r, tail, "\alpha^0"] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^{s-1}"] & X^s \arrow[r, tail, "\alpha^s"] & \cdots \arrow[r, tail, "\alpha^{t-1}"] & X^t \arrow[r, tail, "\alpha^t"] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l.
\end{tikzcd}$$

Due to Lemma 2.17, this yields a distinguished triangle

$$\begin{tikzcd}
\tilde{C}^* \arrow[d] & 0 \arrow[r, equal] \arrow[d] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] \arrow[d] & \tilde{C}^s \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & \tilde{C}^t \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & P(X^{t+1}) \arrow[r, equal] \arrow[d, two heads, "p_{X^{t+1}}"] & \cdots \arrow[r, equal] & P(X^{t+1}) \arrow[d] \\
X \arrow[d, "f"'] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^{s-1}"] \arrow[d, equal] & X^s \arrow[r, tail, "\alpha^s"] \arrow[d, tail, "\alpha^t\cdots\alpha^s"] & \cdots \arrow[r, tail, "\alpha^{t-1}"] & X^t \arrow[r, tail, "\alpha^t"] \arrow[d, tail, "\alpha^t"] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d, equal] \\
\tilde{X} & X^0 \arrow[r, tail, "\alpha^0"] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^t\cdots\alpha^{s-1}"] & X^{t+1} \arrow[r, equal] & \cdots \arrow[r, equal] & X^{t+1} \arrow[r, equal] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] & \cdots \arrow[r, tail, "\alpha^l"] & X^l
\end{tikzcd}$$

in $\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{F})$ with $\tilde{X} = \delta^{[s,t+1]}\gamma^{[s,t]}(X) \in \Delta^{[s,t+1]}$ and $\tilde{C}^* \in \Gamma^{[s,t]}$. Proposition 2.3 yields the claims on adjoints.

(b) For condition (a) in Definition 2.1, consider a morphism $f\colon (X,\alpha) \to (Y,\beta)$ in $\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{F})$ with $X \in \Delta^{[s-1,t]}$ and $Y \in \Gamma^{[s,t]}$. It factors as

$$\begin{tikzcd}
X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, "f^0"] & \cdots \arrow[r, tail, "\alpha^{s-3}"] & X^{s-2} \arrow[r, tail, "\alpha^{s-2}"] \arrow[d, "f^{s-2}"] & X^{s-1} \arrow[r, equal] \arrow[d, "f^{s-1}"] & X^s \arrow[r, equal] \arrow[d, "f^{s-1}"] & \cdots \arrow[r, equal] & X^t \arrow[r, tail, "\alpha^t"] \arrow[d, "f^{s-1}"] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d, "f^{t+1}"] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d, "f^l"] \\
Y^0 \arrow[r, tail, "\beta^0"] \arrow[d, equal] & \cdots \arrow[r, tail, "\beta^{s-3}"] & Y^{s-2} \arrow[r, tail, "\beta^{s-2}"] \arrow[d, equal] & Y^{s-1} \arrow[r, equal] \arrow[d, equal] & Y^{s-1} \arrow[r, equal] \arrow[d, "\beta^{s-1}"] & \cdots \arrow[r, equal] & Y^{s-1} \arrow[r, tail, "\beta^t\cdots\beta^{s-1}"] \arrow[d, "\beta^{t-1}\cdots\beta^{s-1}"] & Y^{t+1} \arrow[r, tail, "\beta^{t+1}"] \arrow[d, equal] & \cdots \arrow[r, tail, "\beta^{l-1}"] & Y^l \arrow[d, equal] \\
Y^0 \arrow[r, tail, "\beta^0"] & \cdots \arrow[r, tail, "\beta^{s-3}"] & Y^{s-2} \arrow[r, tail, "\beta^{s-2}"] & Y^{s-1} \arrow[r, tail, "\beta^{s-1}"] & Y^s \arrow[r, tail, "\beta^s"] & \cdots \arrow[r, tail, "\beta^{t-1}"] & Y^t \arrow[r, tail, "\beta^t"] & Y^{t+1} \arrow[r, tail, "\beta^{t+1}"] & \cdots \arrow[r, tail, "\beta^{l-1}"] & Y^l,
\end{tikzcd}$$

since $\alpha^{k-1} = \mathrm{id}_{X^{k-1}}$ and hence $f^k = \beta^{k-1}\cdots\beta^{s-1}f^{s-1}$ for $k \in \{s, \dots, t\}$. The object in the middle row is projective since $Y^k \in \mathrm{Proj}(\mathcal{F})$ for all $k \in \{0, \dots, s-1, t+1, \dots, l\}$, see Theorem 2.20.(b), and hence $f = 0$ in $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$.

For condition (b) in Definition 2.1, let $X = (X, \alpha) \in \mathrm{Mor}_l^{\mathrm{m}}(\mathcal{F})$ be arbitrary, and set $\tilde{X} := \delta^{[s-1,t]}\gamma^{[s,t]}(X) \in \Delta^{[s-1,t]}$. We prove that the morphism

$$\begin{tikzcd}
\tilde{X} \arrow[d, "f"'] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, equal] \arrow[d, equal] & X^{s-1} \arrow[r, equal] \arrow[d, tail, "\alpha^{s-1}"] & \cdots \arrow[r, equal] & X^{s-1} \arrow[r, tail, "\alpha^t\cdots\alpha^{s-1}"] \arrow[d, tail, "\alpha^{t-1}\cdots\alpha^{s-1}"] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d, equal] \\
X & X^0 \arrow[r, tail, "\alpha^0"] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^{s-1}"] & X^s \arrow[r, tail, "\alpha^s"] & \cdots \arrow[r, tail, "\alpha^{t-1}"] & X^t \arrow[r, tail, "\alpha^t"] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l
\end{tikzcd}$$

in $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ has its cone $C := C(f)$ in $\Gamma^{[s,t]}$. To this end, we consider the pushout of f along the admissible monic $i := i_{\tilde{X}} \colon \tilde{X} \rightarrowtail I(\tilde{X}) := I$ with $i^k = i^{s-1}$ for all $k \in \{s, \dots, t\}$, see Construction 2.23.(b). Due to Remark 2.24, it displays as a termwise pushout:

$$\begin{tikzcd}[column sep=small]
& I \arrow[dd] & & I^0 \arrow[rr, tail] \arrow[dd, equal] & & \cdots \arrow[rr, tail] & & I^{s-1} \arrow[rr, equal] \arrow[dd, equal] & & I^s \arrow[rr, equal] \arrow[dd, tail] & & \cdots \arrow[rr, equal] & & I^t \arrow[rr, tail] \arrow[dd, tail] & & I^{t+1} \arrow[rr, tail] \arrow[dd, equal] & & \cdots \arrow[rr, tail] & & I^l \arrow[dd, equal] \\
\tilde{X} \arrow[ur, tail, "i"] \arrow[dd, "f"'] & & X^0 \arrow[ur, tail] \arrow[rr, tail] \arrow[dd, equal] & & \cdots \arrow[rr, tail] & & X^{s-1} \arrow[ur, tail] \arrow[rr, equal] \arrow[dd, equal] & & X^{s-1} \arrow[ur, tail] \arrow[rr, equal] \arrow[dd, tail] & & \cdots \arrow[rr, equal] & & X^{s-1} \arrow[ur, tail] \arrow[rr, tail] \arrow[dd, tail] & & X^{t+1} \arrow[ur, tail] \arrow[rr, tail] \arrow[dd, equal] & & \cdots \arrow[rr, tail] & & X^l \arrow[ur, tail] \arrow[dd, equal] & \\
& C & & I^0 \arrow[rr, tail] & & \cdots \arrow[rr, tail] & & I^{s-1} \arrow[rr, tail] & & C^s \arrow[rr, tail] & & \cdots \arrow[rr, tail] & & C^t \arrow[rr, tail] & & I^{t+1} \arrow[rr, tail] & & \cdots \arrow[rr, tail] & & I^l \\
X \arrow[ur, tail] & & X^0 \arrow[ur, tail] \arrow[rr, tail] & & \cdots \arrow[rr, tail] & & X^{s-1} \arrow[ur, tail] \arrow[rr, tail] & & X^s \arrow[ur, tail] \arrow[rr, tail] & & \cdots \arrow[rr, tail] & & X^t \arrow[ur, tail] \arrow[rr, tail] & & X^{t+1} \arrow[ur, tail] \arrow[rr, tail] & & \cdots \arrow[rr, tail] & & X^l \arrow[ur, tail] &
\end{tikzcd}$$

In particular, for each $k \in \{s, \dots, t\}$, the square

$$\begin{tikzcd}
X^{s-1} \arrow[r, tail, "\alpha^{k-1}\cdots\alpha^{s-1}"] \arrow[d, tail] \arrow[dr, phantom, "\square"] & X^k \arrow[d, tail] \\
I^{s-1} = I^k \arrow[r, tail] & C^k
\end{tikzcd}$$

is bicartesian and, for $k > s$, so is right square in the diagram

$$\begin{tikzcd}
X^{s-1} \arrow[r, tail, "\alpha^{k-2}\cdots\alpha^{s-1}"] \arrow[d, tail] \arrow[dr, phantom, "\square"] & X^{k-1} \arrow[r, tail, "\alpha^{k-1}"] \arrow[d, tail] \arrow[dr, phantom, "\square"] & X^k \arrow[d, tail] \\
I^{s-1} \arrow[r, tail] & C^{k-1} \arrow[r, tail] & C^k
\end{tikzcd}$$

due to Proposition 2.9.(a) and Lemma 4.6.(a). We postcompose $X \to C$ with the quasi-isomorphism $C \to \tilde{C}$ from Lemma 3.7.(a) and the one from Corollary 4.2 (applied to the identity morphism of $\gamma^{[t+1,l]}(\tilde{C})$ and the monics $\tilde{C}^t \rightarrowtail \tilde{C}^{t+1} =: J \in \mathrm{Proj}(\mathcal{F})$ and $i_{\tilde{C}^t}$) to obtain a quasi-isomorphic cone of f:

$$\begin{tikzcd}
X \arrow[d] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, tail] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^{s-1}"] \arrow[d, tail] \arrow[rd, phantom, "\square"] & X^s \arrow[r, tail, "\alpha^s"] \arrow[d, tail] \arrow[rd, phantom, "\square"] & \cdots \arrow[r, tail, "\alpha^{t-1}"] \arrow[rd, phantom, "\square"] & X^t \arrow[r, tail, "\alpha^t"] \arrow[d, tail] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d, tail] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d, tail] \\
C \arrow[d] & I^0 \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & I^{s-1} \arrow[r, tail] \arrow[d, two heads] \arrow[rd, phantom, "\square"] & C^s \arrow[r, tail] \arrow[d, two heads] \arrow[rd, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[rd, phantom, "\square"] & C^t \arrow[r, tail] \arrow[d, two heads] \arrow[rd, phantom, "\square"] & I^{t+1} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & I^l \arrow[d, two heads] \\
\tilde{C} \arrow[d] & 0 \arrow[r, equal] \arrow[d, equal] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] \arrow[d, equal] \arrow[rd, phantom, "\square"] & \tilde{C}^s \arrow[r, tail] \arrow[d, equal] \arrow[rd, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[rd, phantom, "\square"] & \tilde{C}^t \arrow[r, tail] \arrow[d, equal] & J \arrow[r, equal] \arrow[d] & \cdots \arrow[r, equal] & J \arrow[d] \\
\hat{C} & 0 \arrow[r, equal] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] & \tilde{C}^s \arrow[r, tail] & \cdots \arrow[r, tail] & \tilde{C}^t \arrow[r, tail, "i_{\tilde{C}^t}"] & I(\tilde{C}^t) \arrow[r, equal] & \cdots \arrow[r, equal] & I(\tilde{C}^t).
\end{tikzcd}$$

This yields a distinguished triangle

$$\begin{tikzcd}
\tilde{X} \arrow[d, "f"'] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, equal] \arrow[d, equal] & X^{s-1} \arrow[r, equal] \arrow[d, tail, "\alpha^{s-1}"] & \cdots \arrow[r, equal] & X^{s-1} \arrow[r, tail, "\alpha^t\cdots\alpha^{s-1}"] \arrow[d, tail, "\alpha^{t-1}\cdots\alpha^{s-1}"] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d, equal] \\
X \arrow[d] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^{s-1}"] \arrow[d, two heads] \arrow[rd, phantom, "\square"] & X^s \arrow[r, tail, "\alpha^s"] \arrow[d, two heads] \arrow[rd, phantom, "\square"] & \cdots \arrow[r, tail, "\alpha^{t-1}"] \arrow[rd, phantom, "\square"] & X^t \arrow[r, tail, "\alpha^t"] \arrow[d, two heads] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d] \\
\hat{C} & 0 \arrow[r, equal] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] & \tilde{C}^s \arrow[r, tail] & \cdots \arrow[r, tail] & \tilde{C}^t \arrow[r, tail, "i_{\tilde{C}^t}"] & I(\tilde{C}^t) \arrow[r, equal] & \cdots \arrow[r, equal] & I(\tilde{C}^t)
\end{tikzcd}$$

in $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ with $\tilde{X} = \delta^{[s-1,t]}\gamma^{[s,t]}(X) \in \Delta^{[s-1,t]}$ and $\hat{C} \in \Gamma^{[s,t]}$. Proposition 2.3 yields the claim on adjoints. □

Remark 4.5. There is an alternative distinguished triangle in Proposition 4.4.(b): In the proof, we replace the quasi-isomorphism from Lemma 3.7.(a) by the one obtained from Lemma 3.8 applied to $\gamma^{[0,t]}(C)$. This results in the distinguished triangle

$$\begin{tikzcd}
\tilde{X} \arrow[d] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, equal] \arrow[d, equal] & X^{s-1} \arrow[r, equal] \arrow[d, tail, "\alpha^{s-1}"] & \cdots \arrow[r, equal] & X^{s-1} \arrow[r, tail, "\alpha^t\cdots\alpha^{s-1}"] \arrow[d, tail, "\alpha^{t-1}\cdots\alpha^{s-1}"] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d, equal] \\
X \arrow[d, "g"] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, tail] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^{s-1}"] \arrow[d, tail] \arrow[rd, phantom, "\square"] & X^s \arrow[r, tail, "\alpha^s"] \arrow[d, tail] \arrow[rd, phantom, "\square"] & \cdots \arrow[r, tail, "\alpha^{t-1}"] \arrow[rd, phantom, "\square"] & X^t \arrow[r, tail, "\alpha^t"] \arrow[d, tail] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d] \\
Y & I^0 \arrow[r, tail] & \cdots \arrow[r, tail] & I^{s-1} \arrow[r, tail] & Y^s \arrow[r, tail] & \cdots \arrow[r, tail] & Y^t \arrow[r, tail, "i_{Y^t}"] & I(Y^t) \arrow[r, equal] & \cdots \arrow[r, equal] & I(Y^t)
\end{tikzcd}$$

in $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$, where $\gamma^{[s,l]}(g) = i_{\gamma^{[s,l]}(X)}$. Thus, the inclusion $\Gamma^{[s,t]} \hookrightarrow \underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ has another left adjoint which sends any $X \in \mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$ to Y.

Lemma 4.6 ([FS24, Lem. 1.3])**.** *Consider the following commutative diagram in an additive category:*

$$\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "a"] & B \arrow[r, "s"] \arrow[d, "b"] & C \arrow[d, "c"] \\
A' \arrow[r, "r'"] & B' \arrow[r, "s'"] & C'
\end{tikzcd}$$

(a) If the outer square is a pushout and $\begin{pmatrix} b & r' \end{pmatrix} : B \oplus A' \to B'$ *is an epic, then the right square is a pushout.*

(b) If the outer square is a pullback and $\begin{pmatrix} s \\ b \end{pmatrix} : B \to C \oplus B'$ *is a monic, then the left square is a pullback.* □

Combining Proposition 4.1 and Proposition 4.4 yields

Corollary 4.7. *Let* $\mathcal{F}$ *be a Frobenius category and* $l \in \mathbb{N}_{\geq 1}$. *For each* $s \in \{0, \dots, l-1\}$, *there is the following* $(2l+4)$*-gon of recollements in* $\underline{\mathrm{Mor}}^{\mathrm{m}}_l(\mathcal{F})$*:*

$$\Gamma^{[0,s]}, \dots, \underbrace{\Delta^{[k,k+s+1]}, \Gamma^{[k+1,k+s+1]}}_{for\ k=0,\dots,l-s-1}, \dots, \Gamma^{[0,l-s-1]}, \dots, \underbrace{\Delta^{[k,k+l-s]}, \Gamma^{[k+1,k+l-s]}}_{for\ k=0,\dots,s}$$

$$\begin{tikzcd}
 & \Delta^{[0,s+1]} \arrow[r] & \cdots \arrow[r] & \Gamma^{[k,k+s]} \arrow[r] & \Delta^{[k,k+s+1]} \arrow[r] & \Gamma^{[k+1,k+s+1]} \arrow[r] & \cdots \arrow[r] & \Gamma^{[l-s,l]} \arrow[dr, bend left] & \\
\Gamma^{[0,s]} \arrow[ur, bend left] & & & & & & & & \Gamma^{[0,l-s-1]} \arrow[dl, bend left] \\
 & \Gamma^{[s+1,l]} \arrow[ul, bend left] & \cdots \arrow[l] & \Delta^{[k+1,k+l-s+1]} \arrow[l] & \Gamma^{[k+1,k+l-s]} \arrow[l] & \Delta^{[k,k+l-s]} \arrow[l] & \cdots \arrow[l] & \Delta^{[0,l-s]} \arrow[l] &
\end{tikzcd}$$

If l *is odd and* $s = \frac{l-1}{2}$, *it is invariant under index shift by* $l+2$ *and reduces to the* $(l+2)$*-gon*

$$\Gamma^{[0,s]}, \underbrace{\Delta^{[k,k+s+1]}, \dots, \Gamma^{[k+1,k+s+1]}}_{for\ k=0,\dots,s}.$$ □

5. Mutation and suspension

In this section, we explicitly describe the mutations occurring in the polygon of recollements from Corollary 4.7. We identify the subcategories from Section 3 with smaller stable monomorphism categories using a further type of expansion functors. Under these identifications, certain mutations become the identity functor, while the others agree with one particular, non-trivial auto-equivalence. We show that its $(l+2)$nd power coincides with the square of the suspension functor.

Construction 5.1 (Mutations)**.** The adjoints obtained in Propositions 4.1 and 4.4 combined with the description of $\Gamma^{[s,t]}$ in Corollary 4.3 allow us to explicate the mutations occurring in the polygon from Corollary 4.7, see Definition 2.5:

(a) Due to Propositions 4.4.(b).(iii) and 4.4.(a).(ii), the mutations $L_{\Delta^{[s,t+1]}}$ and $R_{\Delta^{[s,t+1]}}$ are given by

$$\begin{tikzcd}
\Gamma^{[s,t]} \arrow[d, "L_{\Delta^{[s,t+1]}}"', "\simeq"] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] \arrow[d, equal] & X^s \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & X^{s+1} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & X^t \arrow[r, tail, "i_{X^t}"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & I(X^t) \arrow[r, equal] \arrow[d, two heads] & I(X^t) \arrow[r, equal] \arrow[d] & \cdots \\
\Gamma^{[s+1,t+1]} & \cdots \arrow[r, equal] & 0 \arrow[r, equal] & 0 \arrow[r, tail] & Y^{s+1} \arrow[r, tail] & \cdots \arrow[r, tail] & Y^t \arrow[r, tail] & Y^{t+1} \arrow[r, tail, "i_{Y^{t+1}}"] & I(Y^{t+1}) \arrow[r, equal] & \cdots,
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma^{[s,t]} & \cdots \arrow[r, equal] & 0 \arrow[r, tail] \arrow[d, equal] & Y^s \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & Y^{s+1} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & Y^t \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & P(X^{t+1}) \arrow[r, equal] \arrow[d, two heads, "p_{X^{t+1}}"] & P(X^{t+1}) \arrow[r, equal] \arrow[d] & \cdots \\
\Gamma^{[s+1,t+1]} \arrow[u, "R_{\Delta^{[s,t+1]}}", "\simeq"'] & \cdots \arrow[r, equal] & 0 \arrow[r, equal] & 0 \arrow[r, tail] & X^{s+1} \arrow[r, tail] & \cdots \arrow[r, tail] & X^t \arrow[r, tail] & X^{t+1} \arrow[r, tail, "i_{X^{t+1}}"] & I(X^{t+1}) \arrow[r, equal] & \cdots.
\end{tikzcd}$$

(b) Due to Propositions 4.4.(a).(iii) and 4.4.(b).(ii), the mutations $L_{\Gamma^{[s+1,t]}}$ and $R_{\Gamma^{[s+1,t]}}$ are

$$\begin{tikzcd}
\Delta^{[s,t]} \arrow[d, shift right, "L_{\Gamma[s+1,t]}"'] \arrow[d, phantom, "\simeq"] & \cdots \arrow[r, >->] & X^{s-1} \arrow[r, >->] & X^{s} \arrow[r, equal] & X^{s} \arrow[r, equal] & \cdots \arrow[r, equal] & X^{s} \arrow[r, >->] & X^{t+1} \arrow[r, >->] & X^{t+2} \arrow[r, >->] & \cdots \\
\Delta^{[s+1,t+1]} \arrow[u, shift right, "R_{\Gamma[s+1,t]}"'] & \cdots \arrow[r, >->] & X^{s-1} \arrow[r, >->] & X^{s} \arrow[r, >->] & X^{t+1} \arrow[r, equal] & \cdots \arrow[r, equal] & X^{t+1} \arrow[r, equal] & X^{t+1} \arrow[r, >->] & X^{t+2} \arrow[r, >->] & \cdots .
\end{tikzcd}$$

(c) Due to Propositions 4.1.(c) and 4.4.(b).(ii), the mutations $L_{\Gamma[s+1,l]}$ and $R_{\Gamma[s+1,l]}$ are

$$\begin{tikzcd}
\Delta^{[s,l]} \arrow[d, shift right, "L_{\Gamma[s+1,l]}"'] \arrow[d, phantom, "\simeq"] & X^{0} \arrow[r, >->] & \cdots \arrow[r, >->] & X^{s-1} \arrow[r, >->] & X^{s} \arrow[r, equal] & X^{s} \arrow[r, equal] & \cdots \arrow[r, equal] & X^{s} \\
\Gamma^{[0,s]} \arrow[u, shift right, "R_{\Gamma[s+1,l]}"'] & X^{0} \arrow[r, >->] & \cdots \arrow[r, >->] & X^{s-1} \arrow[r, >->] & X^{s} \arrow[r, >->] & I(X^{s}) \arrow[r, equal] & \cdots \arrow[r, equal] & I(X^{s}),
\end{tikzcd}$$

where $R_{\Gamma[s+1,l]}$ is given by $\underline{\delta}^{[s,l]} \circ \underline{\gamma}^{[s+1,l]}$.

(d) Due to Propositions 4.4.(a).(iii) and 4.1.(b), the mutations $L_{\Gamma[0,s-1]}$ and $R_{\Gamma[0,s-1]}$ are

$$\begin{tikzcd}
\Gamma^{[s,l]} \arrow[d, shift right, "L_{\Gamma[0,s-1]}"'] \arrow[d, phantom, "\simeq"] & 0 \arrow[r, >->] & \cdots \arrow[r, >->] & 0 \arrow[r, >->] & X^{s} \arrow[r, >->] & \cdots \arrow[r, >->] & X^{l} \\
\Delta^{[0,s]} \arrow[u, shift right, "R_{\Gamma[0,s-1]}"'] & X^{s} \arrow[r, equal] & \cdots \arrow[r, equal] & X^{s} \arrow[r, equal] & X^{s} \arrow[r, >->] & \cdots \arrow[r, >->] & X^{l},
\end{tikzcd}$$

where $L_{\Gamma[0,s-1]}$ is given by $\underline{\delta}^{[0,s]} \circ \underline{\gamma}^{[0,s-1]}$.

We use the above mutations to identify the subcategory $\Gamma^{[s,t]}$ of $\underline{\mathrm{Mor}}^{\mathrm{m}}{}_{l}(\mathcal{F})$ with $\underline{\mathrm{Mor}}^{\mathrm{m}}{}_{t-s}(\mathcal{F})$:

Construction 5.2 (The expansion functors $\underline{\delta}^{[s,t]^c}$)**.** Let $\mathcal{F}$ be a Frobenius category, $l \in \mathbb{N}$, and $s, t \in \{0, \dots, l\}$ with $s \leq t$.

(a) We pre- and postcompose the left mutation $L_{\Gamma[t+1,l]}$ from Construction 5.1.(c) with the triangle equivalence from Lemma 3.5 given by $\underline{\delta}_t^{[t,l]}$ and the inclusion $\Gamma^{[t,l]} \subseteq \underline{\mathrm{Mor}}^{\mathrm{m}}{}_{l}(\mathcal{F})$ to define a fully faithful triangle functor $\underline{\delta}^{[0,t]^c} := \underline{\delta}_t^{[0,t]^c}$:

$$\begin{tikzcd}
\underline{\mathrm{Mor}}^{\mathrm{m}}{}_{t}(\mathcal{F}) \arrow[r, "\underline{\delta}^{[t,l]}", "\simeq"'] \arrow[rrr, dashed, bend left, "\underline{\delta}_t^{[0,t]^c}"] & \Delta^{[t,l]} \arrow[r, "L_{\Gamma[0,t-1]}", "\simeq"'] & \Gamma^{[0,t]} \arrow[r, hook] \arrow[ll, bend left, "\simeq", "\underline{\gamma}^{[t+1,l]}"'] & \underline{\mathrm{Mor}}^{\mathrm{m}}{}_{l}(\mathcal{F}),
\end{tikzcd}$$

whose quasi-inverse on $\Gamma^{[0,t]}$ is the restriction of $\underline{\gamma}_l^{[t+1,l]}$, since the quasi-inverse $R_{\Gamma[t+1,l]}$ of $L_{\Gamma[t+1,l]}$ is given by $\underline{\delta}^{[t,l]} \circ \underline{\gamma}^{[t+1,l]}$. Explicitly, it sends an object $X \in \mathrm{Mor}^{\mathrm{m}}_{t}(\mathcal{F})$ to

$$X^{0} \rightarrowtail \cdots \rightarrowtail X^{t} \rightarrowtail I(X^{t}) = \cdots = I(X^{t}).$$

(b) We pre- and postcompose the right mutation $R_{\Gamma[0,s-1]}$ from Construction 5.1.(d) with the triangle equivalence from Lemma 3.5 given by $\underline{\delta}_{l-s}^{[0,s]}$ and the inclusion $\Gamma^{[s,l]} \subseteq \underline{\mathrm{Mor}}^{\mathrm{m}}{}_{l}(\mathcal{F})$ to define a fully faithful triangle functor $\underline{\delta}^{[s,l]^c} := \underline{\delta}^{[s,l]^c}_{l-s}$:

$$\begin{tikzcd}
\underline{\mathrm{Mor}^{\mathrm{m}}}_{l-s}(\mathcal{F}) \arrow[r, "\underline{\delta}^{[0,s]}", "\simeq"'] \arrow[rrr, dashed, bend left, "\underline{\delta}^{[s,l]^c}_{l-s}"] & \Delta^{[0,s]} \arrow[r, "R_{\Gamma[0,s-1]}", "\simeq"'] & \Gamma^{[s,l]} \arrow[r, hook] \arrow[ll, bend left, "\simeq", "\underline{\gamma}^{[0,s-1]}"'] & \underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F}),
\end{tikzcd}$$

whose quasi-inverse on $\Gamma^{[s,l]}$ is the restriction of $\underline{\gamma}^{[0,s-1]}_l$, since the quasi-inverse $L_{\Gamma[0,s-1]}$ of $R_{\Gamma[0,s-1]}$ is given by $\underline{\delta}^{[0,s]} \circ \underline{\gamma}^{[0,s-1]}$. Explicitly, it sends an object $X \in \mathrm{Mor}^{\mathrm{m}}_{l-s}(\mathcal{F})$ to

$$0 = \cdots = 0 \rightarrowtail X^0 \rightarrowtail \cdots \rightarrowtail X^{l-s}.$$

Combining (a) and (b), we define the fully faithful triangle functor

$$\underline{\delta}^{[s,t]^c} := \underline{\delta}^{[s,t]^c}_{t-s} := \underline{\delta}^{[t,l]^c}_{t} \circ \underline{\delta}^{[0,s]^c}_{t-s} : \underline{\mathrm{Mor}^{\mathrm{m}}}_{t-s}(\mathcal{F}) \hookrightarrow \underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$$

with image $\Gamma^{[s,t]}$. Explicitly, it sends an object $X \in \mathrm{Mor}^{\mathrm{m}}_{t-s}(\mathcal{F})$ to

$$0 = \cdots = 0 \rightarrowtail X^0 \rightarrowtail \cdots \rightarrowtail X^{t-s} \rightarrowtail I(X^{t-s}) = \cdots = I(X^{t-s}).$$

Its quasi-inverse on $\Gamma^{[s,t]}$ is the restriction of $\underline{\gamma}^{[s,t]^c} = \underline{\gamma}^{[0,s-1]}_t \circ \underline{\gamma}^{[t+1,l]}_l$, see Definition 3.1.(a). We set $\underline{\delta}^{s^c} := \underline{\delta}^{[s,s]^c}$.

Construction 5.3 (Abstract mutations)**.** Let $\mathcal{F}$ be a Frobenius category, $l \in \mathbb{N}$, and $s, t \in \{0, \dots, l\}$ with $s \leq t$. We use the equivalences $\underline{\mathrm{Mor}^{\mathrm{m}}}_{l-t+s}(\mathcal{F}) \xrightarrow{\simeq} \Delta^{[s,t]}$ and $\underline{\mathrm{Mor}^{\mathrm{m}}}_{t-s} \xrightarrow{\simeq} \Gamma^{[s,t]}$ given by $\underline{\delta}^{[s,t]}$ and $\underline{\delta}^{[s,t]^c}$ from Remark 3.3.(a) and Construction 5.2, respectively, as identifications. In this way, we realize the mutations in Construction 5.1 as triangulated auto-equivalences of stable monomorphism categories. While the mutations in (b), (c), and (d) become identity functors, Construction 5.1.(a) yields a non-trivial auto-equivalence:

$$\begin{tikzcd}
& \Gamma^{[s,t]} \arrow[rrr, shift left, "L_{\Delta[s,t+1]}", "\simeq"'] \arrow[dr, hook] & & & \Gamma^{[s+1,t+1]} \arrow[lll, shift left, "R_{\Delta[s,t+1]}"] \arrow[dll, hook'] \\
& & \underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F}) & & \\
\underline{\mathrm{Mor}^{\mathrm{m}}}_{t-s}(\mathcal{F}) \arrow[uur, "\simeq"] \arrow[urr, "\delta^{[s,t]^c}"'] \arrow[rrr, dashed, shift left, "\Theta", "\simeq"'] & & & \underline{\mathrm{Mor}^{\mathrm{m}}}_{t-s}(\mathcal{F}) \arrow[ul, "\delta^{[s+1,t+1]^c}"] \arrow[uur, "\simeq"'] \arrow[lll, dashed, shift left, "\Theta^{-1}"] &
\end{tikzcd}$$

Given $X, Y \in \mathrm{Mor}^{\mathrm{m}}_{t-s}(\mathcal{F})$, it is determined by the diagram

$$\begin{tikzcd}
\Theta^{-1} Y = X & X^0 \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & X^1 \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & X^{t-s} \arrow[r, tail, "i"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & P \arrow[d, two heads, "p"] \\
\Theta X = Y & 0 \arrow[r, tail] & Y^0 \arrow[r, tail] & \cdots \arrow[r, tail] & Y^{t-s-1} \arrow[r, tail, two heads] & Y^{t-s}
\end{tikzcd}$$

of bicartesian squares in $\mathcal{F}$ with $i = i_{X^{t-s}}$ to define Θ and $p = p_{Y^{t-s}}$ to define Θ^{-1}.

Remark 5.4. Let $\mathcal{F}$ be a Frobenius category and $l \in \mathbb{N}$. Using the left adjoint from Remark 4.5 to define the left mutation $L_{\Delta[s,t+1]}$ in Construction 5.1.(a), Construction 5.3 yields a functor $\tilde{\Theta}\colon \underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F}) \to \underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$, isomorphic to Θ, determined by the diagram

$$\begin{tikzcd}
\tilde{\Theta}^{-1}Y = X & X^0 \arrow[r, tail] \arrow[d, tail, "i_{X^0}"'] \arrow[dr, phantom, "\square"] & X^1 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & X^l \arrow[r, tail, "i_{X^l}"] \arrow[d, tail] \arrow[dr, phantom, "\square"] & I(X^l) \arrow[d, tail] \\
\tilde{\Theta}X = Y & I(X^0) \arrow[r, tail] & Y^0 \arrow[r, tail] & \cdots \arrow[r, tail] & Y^{l-1} \arrow[r, tail] & Y^l
\end{tikzcd}$$

of bicartesian squares in $\mathcal{F}$. For any object $X \in \mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$, Θ and $\tilde{\Theta}$ are related by $\tilde{\Theta}X \cong \Theta X \oplus \mu_{l+1}(I(X^0))$, see (3.1).

Proposition 5.5. *For any Frobenius category $\mathcal{F}$ and all $l \in \mathbb{N}$, there is an isomorphism $\Theta^{l+2} \cong \Sigma^2$ of endofunctors on $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$, where Σ is the suspension functor.*

Proof. We prove instead that $\tilde{\Theta}^{l+2} \cong \Sigma^2$, see Remark 5.4. For an arbitrary $X \in \mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$, set $X_0^k := X^k$ for $k \in \{0, \ldots, l\}$. The $(l+2)$-fold application of $\tilde{\Theta}$, is given by a diagram

$$\begin{tikzcd}[column sep=small]
X_0^0 \arrow[r, tail] \arrow[d, tail, "i_{X_0^0}"] \arrow[dr, phantom, "\square"] & X_0^1 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & X_0^2 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & X_0^l \arrow[r, tail, "i_{X_0^l}"] \arrow[d, tail] \arrow[dr, phantom, "\square"] & J_0 \arrow[d, tail] \arrow[dr, dashed, tail] & & & & & \\
I_1 \arrow[r, tail] \arrow[dr, dashed, tail] & X_1^0 \arrow[r, tail] \arrow[d, tail, "i_{X_1^0}"] \arrow[dr, phantom, "\square"] & X_1^1 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & X_1^{l-1} \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & X_1^l \arrow[r, tail, "i_{X_1^l}"] \arrow[d, tail] \arrow[dr, phantom, "\square"] & J_1 \arrow[d, tail] \arrow[dr, dashed, tail] & & & & \\
& I_2 \arrow[r, tail] \arrow[dr, dashed, tail] & X_2^0 \arrow[r, tail] & \cdots \arrow[r, tail] & X_2^{l-2} \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & X_2^{l-1} \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & X_2^l \arrow[r, tail, "i_{X_2^l}"] \arrow[d, tail] \arrow[dr, phantom, "\square"] & J_2 \arrow[d, tail] \arrow[dr, dashed, tail] & & & \\
& & \ddots \arrow[dr, dashed, tail] & & \vdots \arrow[d, tail] \arrow[dr, phantom, "\square"] & \vdots \arrow[d, tail] \arrow[dr, phantom, "\square"] & \vdots \arrow[d, tail] \arrow[dr, phantom, "\square"] & \vdots \arrow[d, tail] & \ddots \arrow[dr, dashed, tail] & & \\
& & & I_l \arrow[r, tail] \arrow[dr, dashed, tail] & X_l^0 \arrow[r, tail] \arrow[d, tail, "i_{X_l^0}"] \arrow[dr, phantom, "\square"] & X_l^1 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & X_l^2 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & X_l^3 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail, "i_{X_l^l}"] \arrow[dr, phantom, "\square"] & J^l \arrow[d, tail] \arrow[dr, dashed, tail] & \\
& & & & I_{l+1} \arrow[r, tail] \arrow[dr, dashed, tail] & X_{l+1}^0 \arrow[r, tail] \arrow[d, tail, "i_{X_{l+1}^0}"] \arrow[dr, phantom, "\square"] & X_{l+1}^1 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & X_{l+1}^2 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & X_{l+1}^l \arrow[r, tail, "i_{X_{l+1}^l}"] \arrow[d, tail] \arrow[dr, phantom, "\square"] & J^{l+1} \arrow[d, tail] \\
& & & & & I_{l+2} \arrow[r, tail] & X_{l+2}^0 \arrow[r, tail] & X_{l+2}^1 \arrow[r, tail] & \cdots \arrow[r, tail] & X_{l+2}^{l-1} \arrow[r, tail] & X_{l+2}^l
\end{tikzcd} \tag{5.1}$$

of bicartesian squares in $\mathcal{F}$, where $J^j := I(X_j^l)$, $I^{j+1} := I(X_j^0)$ and $X_j = \tilde{\Theta}^j X_0$ for $j \in \{0, \ldots, l+2\}$. Consider the following objects of $\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$:

$$Y\colon\ X_1^l \rightarrowtail X_2^{l-1} \rightarrowtail \cdots \rightarrowtail X_{l+1}^0,$$

$$I\colon\ I^1 \rightarrowtail I^2 \rightarrowtail \cdots \rightarrowtail I^{l+1},$$

$$J\colon\ J^1 \rightarrowtail J^2 \rightarrowtail \cdots \rightarrowtail J^{l+1}.$$

Note that $I = I(X)$ and $J = I(Y)$ and that the respective morphisms $X \to I$ and $Y \to J$ defined by (5.1) agree with i_X and i_Y, respectively, see Construction 2.23.(b). Using Lemma 5.6, (5.1) yields short exact sequences

$$\begin{tikzcd}
X \arrow[r, tail, "\begin{pmatrix} i_X \\ * \end{pmatrix}"] & I(X) \oplus \mu_{l+1}(J_0) \arrow[r, two heads] & Y,
\end{tikzcd}$$

$$\begin{tikzcd}
Y \arrow[r, tail, "\begin{pmatrix} i_Y \\ * \end{pmatrix}"] & I(Y) \oplus \mu_{l+1}(I_{l+2}) \arrow[r, two heads] & \tilde{\Theta}^{l+2} X
\end{tikzcd}$$

in $\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{F})$, where $*$ denotes unspecified morphisms. As a consequence, we obtain isomorphisms $Y \cong \Sigma X$ and $\tilde{\Theta}^{l+2} X \cong \Sigma Y \cong \Sigma^2 X$ in $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$, functorial in X, see Construction 2.14. □

Lemma 5.6. *In an exact category $\mathcal{E}$, consider any diagram*

$$\begin{tikzcd}
A \arrow[r, tail, "a"] \arrow[d, tail, "i"] \arrow[dr, phantom, "\square"] & A' \arrow[r, tail, "a'"] \arrow[d, tail, "i'"] \arrow[dr, phantom, "\square"] & A'' \arrow[d, tail, "i''"] \\
B \arrow[r, tail, "b"] & B' \arrow[r, tail, "b'"] \arrow[d, tail, "j'"] \arrow[dr, phantom, "\square"] & B'' \arrow[d, tail, "j''"] \\
& C' \arrow[r, tail, "c'"] & C''
\end{tikzcd}$$

of bicartesian squares gives rise to a short exact sequence

$$\begin{tikzcd}
A \arrow[r, tail, "\begin{pmatrix} a'a \\ -i \end{pmatrix}"] \arrow[d, tail, "a"] & A'' \oplus B \arrow[r, two heads, "\begin{pmatrix} i'' & b'b \end{pmatrix}"] \arrow[d, tail, "\begin{pmatrix} \mathrm{id}_A & 0 \\ 0 & j'b \end{pmatrix}"] & B'' \arrow[d, tail, "j''"] \\
A' \arrow[r, tail, "\begin{pmatrix} a' \\ -j'i' \end{pmatrix}"] & A'' \oplus C' \arrow[r, two heads, "\begin{pmatrix} j''i'' & c' \end{pmatrix}"] & C''
\end{tikzcd}$$

in $\mathrm{Mor}_1^{\mathrm{m}}(\mathcal{E})$.

Proof. The horizontal short exact sequences in $\mathcal{E}$ arise from Proposition 2.9.(a), since concatenation preserves bicartesian squares due to the pasting laws, see [Mac98, Ex. III.4.8]. The middle morphism is an admissible monic due to Proposition 2.8. Commutativity can be checked easily. □

6. Infinite adjoint chains

In this section, we establish adjunctions between the contraction and expansion functors from Sections 3 and 5. In order to form infinite adjoint chains, we introduce two further types of contraction functors. The triangles realizing the semiorthogonal decompositions in Section 4 can be expressed in terms of the constructed functors.

Lemma 6.1. *For any Frobenius category $\mathcal{F}$, $l \in \mathbb{N}$, and $s, t \in \{0, \dots, l\}$ with $s < t$, there are the following pairs of adjoint functors:*

- *(a)* $\big(\underline{\gamma}^{[s,t-1]}, \underline{\delta}^{[s,t]}\big)$ *between the categories* $\underline{\mathrm{Mor}^{\mathrm{m}}}_{l-t+s}(\mathcal{F})$ *and* $\underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$*,*
- *(b)* $\big(\underline{\delta}^{[s,t]}, \underline{\gamma}^{[s+1,t]}\big)$ *between the categories* $\underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$ *and* $\underline{\mathrm{Mor}^{\mathrm{m}}}_{l-t+s}(\mathcal{F})$*,*
- *(c)* $\big(\underline{\gamma}^{[t,l]}, \underline{\delta}^{[0,t-1]^c}\big)$ *between the categories* $\underline{\mathrm{Mor}^{\mathrm{m}}}_{t-1}(\mathcal{F})$ *and* $\underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$*,*
- *(d)* $\big(\underline{\delta}^{[s+1,l]^c}, \underline{\gamma}^{[0,s]}\big)$ *between the categories* $\underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$ *and* $\underline{\mathrm{Mor}^{\mathrm{m}}}_{l-s-1}(\mathcal{F})$*.*

Proof. For (a) and (b), note that the left adjoint of the embedding $\Delta^{[s,t]} \hookrightarrow \underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$ is given by $\underline{\delta}^{[s,t]} \circ \underline{\gamma}^{[s,t-1]}$, the right adjoint by $\underline{\delta}^{[s,t]} \circ \underline{\gamma}^{[s+1,t]}$, see Proposition 4.4.(a).(iii) and (b).(ii). Together with the equivalence from Lemma 3.5 and its quasi-inverse, they fit in commutative diagrams of adjoint functors

$$\begin{tikzcd}
\underline{\mathrm{Mor}^{\mathrm{m}}}_{l-t+s}(\mathcal{F}) \arrow[r, "\simeq"] \arrow[rr, bend right, "\underline{\delta}^{[s,t]}"'] & \Delta^{[s,t]} \arrow[r, hook] \arrow[l, bend right, "\simeq"'] & \underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F}), \arrow[l, bend right, "\underline{\delta}^{[s,t]} \circ \underline{\gamma}^{[s,t-1]}"'] \arrow[ll, bend right=60, "\underline{\gamma}^{[s,t-1]}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{\mathrm{Mor}^{\mathrm{m}}}_{l-t+s}(\mathcal{F}) \arrow[r, "\simeq"] \arrow[rr, bend left, "\underline{\delta}^{[s,t]}"] & \Delta^{[s,t]} \arrow[r, hook] \arrow[l, bend left, "\simeq"] & \underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F}). \arrow[l, bend left, "\underline{\delta}^{[s,t]} \circ \underline{\gamma}^{[s+1,t]}"] \arrow[ll, bend left=60, "\underline{\gamma}^{[s+1,t]}"]
\end{tikzcd}$$

By composition, this yields the desired pairs of adjoint functors. For (c) and (d), note that the left adjoint of $\Gamma^{[0,t-1]} \hookrightarrow \underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$ is given by $\underline{\delta}^{[0,t-1]^c} \circ \underline{\gamma}^{[t,l]}$, the right adjoint of $\Gamma^{[s+1,l]} \hookrightarrow \underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$ by $\underline{\delta}^{[s+1,l]^c} \circ \underline{\gamma}^{[0,s]}$, see Proposition 4.1.(b) and (c) and Construction 5.2. We obtain the following commutative diagrams of adjoint functors, which yield the claim by composition:

$$\begin{tikzcd}
\underline{\mathrm{Mor}^{\mathrm{m}}}_{t-1}(\mathcal{F}) \arrow[r, "\simeq"] \arrow[rr, bend right, "\underline{\delta}^{[0,t-1]^c}"'] & \Gamma^{[0,t-1]} \arrow[r, hook] \arrow[l, bend right, "\simeq"'] & \underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F}) \arrow[l, bend right, "\underline{\delta}^{[0,t-1]^c} \circ \underline{\gamma}^{[t,l]}"'] \arrow[ll, bend right=60, "\underline{\gamma}^{[t,l]}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{\mathrm{Mor}^{\mathrm{m}}}_{l-s-1}(\mathcal{F}) \arrow[r, "\simeq"] \arrow[rr, bend left, "\underline{\delta}^{[s+1,l]^c}"] & \Gamma^{[s+1,l]} \arrow[r, hook] \arrow[l, bend left, "\simeq"] & \underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F}) \arrow[l, bend left, "\underline{\delta}^{[s+1,l]^c} \circ \underline{\gamma}^{[0,s]}"] \arrow[ll, bend left=60, "\underline{\gamma}^{[0,s]}"]
\end{tikzcd}$$

□

We need two more functors to form an infinite adjoint chain:

Construction 6.2 (The contraction functors $\hat{\underline{\gamma}}^{[s,t]^c}$ and $\check{\underline{\gamma}}^{[s,t]^c}$)**.** Let $\mathcal{F}$ be a Frobenius category, $l \in \mathbb{N}$ and $s, t \in \{0, \dots, l\}$ with $s \leq t$.

(a) Postcomposing the left adjoint of $\Gamma^{[s,t]} \hookrightarrow \underline{\mathrm{Mor}}^{\mathrm{m}}{}_l(\mathcal{F})$, see Proposition 4.4.(b).(iii), with the triangle functor $\underline{\gamma}^{[s,t]^c}$ restricted to $\Gamma^{[s,t]}$, see Construction 5.2, yields a triangle functor $\hat{\underline{\gamma}}^{[s,t]^c} := \hat{\underline{\gamma}}_l^{[s,t]^c}$:

$$\begin{tikzcd}
\underline{\mathrm{Mor}}^{\mathrm{m}}{}_{t-s}(\mathcal{F}) \arrow[r, "\simeq"'] \arrow[rr, bend right=20, "\underline{\delta}^{[s,t]^c}"'] & \Gamma^{[s,t]} \arrow[r, hook] \arrow[l, bend right, "\underline{\gamma}^{[s,t]^c}"', "\simeq"] & \underline{\mathrm{Mor}}^{\mathrm{m}}{}_l(\mathcal{F}), \arrow[l, bend right] \arrow[ll, dashed, bend right=50, "\hat{\underline{\gamma}}_l^{[s,t]^c}"']
\end{tikzcd}$$

It sends an object $X \in \mathrm{Mor}_l^{\mathrm{m}}(\mathcal{F})$ to $Y \in \mathrm{Mor}_{t-s}^{\mathrm{m}}(\mathcal{F})$ given by the following commutative diagram:

$$\begin{tikzcd}
X^{s-1} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & X^s \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & X^t \arrow[d, two heads] \\
0 \arrow[r, tail] & Y^0 \arrow[r, tail] & \cdots \arrow[r, tail] & Y^{t-s}
\end{tikzcd}$$

We set $\hat{\underline{\gamma}}^{s^c} := \hat{\underline{\gamma}}_l^{s^c} := \hat{\underline{\gamma}}_l^{[s,s]^c}$.

(b) Postcomposing the right adjoint of $\Gamma^{[s,t]} \hookrightarrow \underline{\mathrm{Mor}}^{\mathrm{m}}{}_l(\mathcal{F})$, see Proposition 4.4.(a).(ii), with the triangle functor $\underline{\gamma}^{[s,t]^c}$ restricted to $\Gamma^{[s,t]}$, see Construction 5.2, yields a triangle functor $\check{\underline{\gamma}}^{[s,t]^c} := \check{\underline{\gamma}}_l^{[s,t]^c}$:

$$\begin{tikzcd}
\underline{\mathrm{Mor}}^{\mathrm{m}}{}_{t-s}(\mathcal{F}) \arrow[r, "\simeq"] \arrow[rr, bend left=20, "\underline{\delta}^{[s,t]^c}"] & \Gamma^{[s,t]} \arrow[r, hook] \arrow[l, bend left, "\underline{\gamma}^{[s,t]^c}", "\simeq"'] & \underline{\mathrm{Mor}}^{\mathrm{m}}{}_l(\mathcal{F}), \arrow[l, bend left] \arrow[ll, dashed, bend left=50, "\check{\underline{\gamma}}_l^{[s,t]^c}"]
\end{tikzcd}$$

It sends an object $X \in \mathrm{Mor}_l^{\mathrm{m}}(\mathcal{F})$ to $Y \in \mathrm{Mor}_{t-s}^{\mathrm{m}}(\mathcal{F})$ given by the following commutative diagram:

$$\begin{tikzcd}
Y^0 \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & Y^{t-s} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & P(X^{t+1}) \arrow[d, two heads, "p_{X^{t+1}}"] \\
X^s \arrow[r, tail] & \cdots \arrow[r, tail] & X^t \arrow[r, tail] & X^{t+1}
\end{tikzcd}$$

We set $\check{\underline{\gamma}}^{s^c} := \check{\underline{\gamma}}_l^{s^c} := \check{\underline{\gamma}}_l^{[s,s]^c}$.

Lemma 6.3. *For any Frobenius category $\mathcal{F}$, $l \in \mathbb{N}$ and $s, t \in \{0, \dots, l\}$ with $s < t$, there are pairs of adjoint functors*

(a) $\left(\hat{\underline{\gamma}}_l^{[s,t]^c}, \underline{\delta}_{t-s}^{[s,t]^c}\right)$ *between the categories* $\underline{\mathrm{Mor}}^{\mathrm{m}}{}_{t-s}(\mathcal{F})$ *and* $\underline{\mathrm{Mor}}^{\mathrm{m}}{}_l(\mathcal{F})$*,*

(b) $\left(\underline{\delta}_{t-s}^{[s,t]^c}, \check{\underline{\gamma}}_l^{[s,t]^c}\right)$ *between the categories* $\underline{\mathrm{Mor}}^{\mathrm{m}}{}_l(\mathcal{F})$ *and* $\underline{\mathrm{Mor}}^{\mathrm{m}}{}_{t-s}(\mathcal{F})$*.*

Proof. This is immediate from Construction 6.2. □

Lemma 6.4. *For any Frobenius category $\mathcal{F}$, $l \in \mathbb{N}$ and $s, t \in \{1, \dots, l-1\}$ with $s \leq t$, there are the pairs of adjoint functors* *(a)* $\left(\underline{\delta}_{t-s}^{[s-1,t-1]^c} \circ \Theta^{-1}, \hat{\underline{\gamma}}_l^{[s,t]^c}\right)$ *between the categories* $\underline{\mathrm{Mor}^{\mathrm{m}}}_{t-s}(\mathcal{F})$ *and* $\underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$*,*

(b) $\left(\check{\underline{\gamma}}_l^{[s,t]^c}, \underline{\delta}_{t-s}^{[s+1,t+1]^c} \circ \Theta\right)$ *between the categories* $\underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F})$ *and* $\underline{\mathrm{Mor}^{\mathrm{m}}}_{t-s}(\mathcal{F})$*.*

Proof. Due to Construction 5.3 and Remark 2.6, we have the following commutative diagrams of adjoint functors, see Construction 5.2:

$$\begin{tikzcd}
& \underline{\mathrm{Mor}^{\mathrm{m}}}_{t-s}(\mathcal{F}) \arrow[rr, "\simeq"] \arrow[drrr, bend left=40, "\underline{\delta}^{[s+1,t+1]^c}"] & & \Gamma^{[s-1,t-1]} \arrow[dr, hook] & \\
\underline{\mathrm{Mor}^{\mathrm{m}}}_{t-s}(\mathcal{F}) \arrow[ur, "\Theta^{-1}", "\simeq"'] \arrow[rr, "\underline{\delta}^{[s,t]^c}", "\simeq"'] & & \Gamma^{[s,t]} \arrow[ur, "R_{\Delta[s-1,t]}", "\simeq"'] \arrow[ll, bend left=30, "\underline{\gamma}^{[s,t]^c}", "\simeq"'] \arrow[rr, hook] \arrow[rr, bend left=30] & & \underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F}), \arrow[ll, bend right=15] \arrow[llll, bend left=35, "\hat{\underline{\gamma}}^{[s,t]^c}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{\mathrm{Mor}^{\mathrm{m}}}_{t-s}(\mathcal{F}) \arrow[rr, "\simeq", "\underline{\delta}^{[s,t]^c}"'] \arrow[dr, "\simeq", "\Theta"'] & & \Gamma^{[s,t]} \arrow[ll, bend right=30, "\underline{\gamma}^{[s,t]^c}"', "\simeq"] \arrow[rr, hook] \arrow[rr, bend right=30] \arrow[dr, "\simeq", "L_{\Delta[s-1,t]}"'] & & \underline{\mathrm{Mor}^{\mathrm{m}}}_{l}(\mathcal{F}). \arrow[ll, bend left=15] \arrow[llll, bend right=35, "\check{\underline{\gamma}}^{[s,t]^c}"'] \\
& \underline{\mathrm{Mor}^{\mathrm{m}}}_{t-s}(\mathcal{F}) \arrow[rr, "\simeq"'] \arrow[urrr, bend right=40, "\underline{\delta}^{[s+1,t+1]^c}"'] & & \Gamma^{[s+1,t+1]} \arrow[ur, hook] &
\end{tikzcd}$$

□

Combining Lemmas 6.1, 6.3 and 6.4 we obtain

Theorem 6.5. *Let* $\mathcal{F}$ *be a Frobenius category* $\mathcal{F}$*,* $l \in \mathbb{N}$*, and* $s, t \in \{0, \ldots, l\}$ *with* $s \leq t$*. There is the following infinite adjoint chain:*

$$\cdots \dashv \Theta^2 \hat{\underline{\gamma}}^{[t-s-1,l-2]^c} \dashv \underline{\delta}^{[t-s-1,l-2]^c}\Theta^{-2} \dashv \Theta\hat{\underline{\gamma}}^{[t-s,l-1]^c} \dashv \underline{\delta}^{[t-s,l-1]^c}\Theta^{-1} \dashv \hat{\underline{\gamma}}^{[t-s+1,l]^c} \dashv \underline{\delta}^{[t-s+1,l]^c}$$
$$\dashv \underline{\gamma}^{[0,t-s]} \dashv \cdots \dashv \underline{\gamma}^{[s-1,t-1]} \dashv \underline{\delta}^{[s-1,t]} \dashv \underline{\gamma}^{[s,t]} \dashv \underline{\delta}^{[s,t+1]} \dashv \underline{\gamma}^{[s+1,t+1]} \dashv \cdots \dashv \underline{\gamma}^{[l-t+s,l]}$$
$$\dashv \underline{\delta}^{[0,l-t+s-1]^c} \dashv \check{\underline{\gamma}}^{[0,l-t+s-1]^c} \dashv \underline{\delta}^{[1,l-t+s]^c}\Theta \dashv \Theta^{-1}\check{\underline{\gamma}}^{[1,l-t+s]^c} \dashv \underline{\delta}^{[2,l-t+s+1]^c}\Theta^2 \dashv \Theta^{-2}\check{\underline{\gamma}}^{[2,l-t+s+1]^c} \dashv \cdots$$

□

Using Corollary 4.2 and the functors from Constructions 5.2 and 6.2, we can interpret the distinguished triangles in Propositions 4.1 and 4.4 in the language of Proposition 2.3:

Corollary 6.6. *Let $\mathcal{F}$ be a Frobenius category, $l \in \mathbb{N}$ and $s, t \in \{0, \dots, l\}$ with $s \leq t$. Any $X \in \underline{\mathrm{Mor}}^{\mathrm{m}}_l(\mathcal{F})$ fits into distinguished triangles as follows:*

(a) For $s < l$, we have

$$\begin{tikzcd}
\underline{\delta}^{[s+1,l]^c}\underline{\gamma}^{[0,s]}(X) \arrow[d] & 0 \arrow[r, equal] \arrow[d] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] \arrow[d] & X^{s+1} \arrow[r, tail, "\alpha^{s+1}"] \arrow[d, equal] & X^{s+2} \arrow[r, tail, "\alpha^{s+2}"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d, equal] \\
X \arrow[d] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{s-1}"] & X^s \arrow[r, tail, "\alpha^s"] \arrow[d, equal] & X^{s+1} \arrow[r, tail, "\alpha^{s+1}"] \arrow[d] & X^{s+2} \arrow[r, tail, "\alpha^{s+2}"] \arrow[d] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d] \\
\underline{\delta}^{[0,s]^c}\underline{\gamma}^{[s+1,l]}(X) & X^0 \arrow[r, tail, "\alpha^0"] & \cdots \arrow[r, tail, "\alpha^{s-1}"] & X^s \arrow[r, tail, "i_{X^s}"] & I(X^s) \arrow[r, equal] & I(X^s) \arrow[r, equal] & \cdots \arrow[r, equal] & I(X^s).
\end{tikzcd}$$

(b) For $t < l$, we have

$$\begin{tikzcd}
\underline{\delta}^{[s,t]^c}\underline{\check{\gamma}}^{[s,t]^c}(X) \arrow[d] & 0 \arrow[r, equal] \arrow[d, tail] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] \arrow[d, tail] & Y^s \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[dr, phantom, "\square"] & Y^t \arrow[r, tail, "i_{Y^t}"] \arrow[d, two heads] & I(Y^t) \arrow[r, equal] \arrow[d] & \cdots \arrow[r, equal] & I(Y^t) \arrow[d] \\
X \arrow[d] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^{s-1}"] \arrow[d, equal] & X^s \arrow[r, tail, "\alpha^s"] \arrow[d, tail, "\alpha^t\cdots\alpha^s"] & \cdots \arrow[r, tail, "\alpha^{t-1}"] & X^t \arrow[r, tail, "\alpha^t"] \arrow[d, tail, "\alpha^t"] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d, equal] \\
\underline{\delta}^{[s,t+1]}\underline{\gamma}^{[s,t]}(X) & X^0 \arrow[r, tail, "\alpha^0"] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail] & X^{t+1} \arrow[r, equal] & \cdots \arrow[r, equal] & X^{t+1} \arrow[r, equal] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l,
\end{tikzcd}$$

where Y^t is defined by

$$\begin{tikzcd}
Y^t \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & P(X^{t+1}) \arrow[d, two heads, "p_{X^{t+1}}"] \\
X^t \arrow[r, tail, "\alpha^t"] & X^{t+1}.
\end{tikzcd}$$

(c) For $0 < s$, we have

$$\begin{tikzcd}
\underline{\delta}^{[s-1,t]}\underline{\gamma}^{[s,t]}(X) \arrow[d] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, equal] \arrow[d, equal] & X^{s-1} \arrow[r, equal] \arrow[d, tail, "\alpha^{s-1}"] & \cdots \arrow[r, equal] & X^{s-1} \arrow[r, tail] \arrow[d, tail, "\alpha^{t-1}\cdots\alpha^{s-1}"] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d, equal] \\
X \arrow[d] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, two heads] & \cdots \arrow[r, tail, "\alpha^{s-2}"] & X^{s-1} \arrow[r, tail, "\alpha^{s-1}"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & X^s \arrow[r, tail, "\alpha^s"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail, "\alpha^{t-1}"] \arrow[dr, phantom, "\square"] & X^t \arrow[r, tail, "\alpha^t"] \arrow[d, two heads] & X^{t+1} \arrow[r, tail, "\alpha^{t+1}"] \arrow[d] & \cdots \arrow[r, tail, "\alpha^{l-1}"] & X^l \arrow[d] \\
\underline{\delta}^{[s,t]^c}\underline{\hat{\gamma}}^{[s,t]^c}(X) & 0 \arrow[r, equal] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] & Y^s \arrow[r, tail] & \cdots \arrow[r, tail] & Y^t \arrow[r, tail, "i_{Y^t}"] & I(Y^t) \arrow[r, equal] & \cdots \arrow[r, equal] & I(Y^t).
\end{tikzcd}$$

□

7. DUALIZED HOM-FUNCTORS

In this section, we review the construction of Bondal and Kapranov to lift representations of cohomological functors from semiorthogonal decompositions. Based on the decompositions from Section 4, we lift representations of dualized hom-functors from an algebraic triangulated category to the larger stable monomorphism categories.

Notation 7.1. Let $\mathcal{T}$ be a category, linear over a field K, and denote the category of K-vector spaces by Vect.

(a) For objects $X, Y \in \mathcal{T}$, we abbreviate

$$h_X := h_X^{\mathcal{T}} := \mathrm{Hom}_{\mathcal{T}}(-, X), \quad h^X := h^X_{\mathcal{T}} := \mathrm{Hom}_{\mathcal{T}}(X, -), \quad \text{and} \quad h^X_Y := \mathrm{Hom}_{\mathcal{T}}(X, Y).$$

(b) Consider a contravariant linear functor $h\colon \mathcal{T} \to \mathrm{Vect}$, an object $X \in \mathcal{T}$, and the set $\mathrm{Nat}(h_X, h)$ of natural transformations. The Yoneda lemma yields a bijection

$$\begin{aligned} \mathrm{Nat}(h_X, h) &\longleftrightarrow h(X), \\ \eta &\longmapsto \eta_X(\mathrm{id}_X) =: e_\eta, \\ \big(h^Y_X \ni f \mapsto h(f)(e) \in h(Y)\big)_{Y \in \mathcal{T}} &\longmapsfrom e. \end{aligned}$$

Note that any such natural transformation is automatically (component-wise) linear.

(c) We abbreviate $(-)^* := \mathrm{Hom}_K(-, K)$. Given a natural transformation $\eta_X\colon h^{\mathcal{T}}_{\tilde{X}} \to \big(h^X_{\mathcal{T}}\big)^*$ of contravariant functors $\mathcal{T} \to \mathrm{Vect}$, where $X, \tilde{X} \in \mathcal{T}$, we denote $e_X := e_{\eta_X} \in \big(h^X_{\tilde{X}}\big)^*$, see (b).

We include the following statement for lack of reference:

Lemma 7.2. *Let $\mathcal{T}$ be a triangulated category, linear over a field. Suppose that there are isomorphisms of functors*

$$\eta_X\colon h^{\mathcal{T}}_{\tilde{X}} \cong \big(h^X_{\mathcal{T}}\big)^* \quad \textit{and} \quad \eta_Y\colon h^{\mathcal{T}}_{\tilde{Y}} \cong \big(h^Y_{\mathcal{T}}\big)^*,$$

where $X, Y, \tilde{X}, \tilde{Y} \in \mathcal{T}$. Then each $f \in \mathrm{Hom}_{\mathcal{T}}(X, Y)$ defines a unique $\tilde{f} \in \mathrm{Hom}_{\mathcal{T}}(\tilde{X}, \tilde{Y})$ such that $e_X(- \circ f) = e_Y(\tilde{f} \circ -)$ on $\mathrm{Hom}_{\mathcal{T}}(Y, \tilde{X})$, see Notation 7.1.(c). This assignment is compatible with compositions.

Proof. By the Yoneda lemma, any morphism $f \in \mathrm{Hom}_{\mathcal{T}}(X, Y)$ corresponds to a unique morphism $\tilde{f} \in \mathrm{Hom}_{\mathcal{T}}(\tilde{X}, \tilde{Y})$ via the following commutative diagram of natural transformations:

$$\begin{tikzcd}
h^{\mathcal{T}}_{\tilde{X}} \arrow[r, "\eta_X", "\cong"'] \arrow[d, dashed, "h^{\mathcal{T}}_{\tilde{f}}"] & \big(h^X_{\mathcal{T}}\big)^* \arrow[d, "\big(h^f_{\mathcal{T}}\big)^*"] & \mathrm{id}_{\tilde{X}} \arrow[r, leftrightarrow] \arrow[d, mapsto] & e_X \arrow[d, mapsto] \\
h^{\mathcal{T}}_{\tilde{Y}} \arrow[r, "\eta_Y", "\cong"'] & \big(h^Y_{\mathcal{T}}\big)^* & \tilde{f} \arrow[r, leftrightarrow] & e_Y(\tilde{f} \circ -) = e_X(- \circ f)
\end{tikzcd}$$

This yields the first claim. For the second, consider another natural transformation $\eta_Z\colon h^{\mathcal{T}}_{\tilde{Z}} \cong \big(h^Z_{\mathcal{T}}\big)^*$, where $Z, \tilde{Z} \in \mathcal{T}$, $g \in \mathrm{Hom}_{\mathcal{T}}(Y, Z)$, and $\tilde{g} \in \mathrm{Hom}_{\mathcal{T}}(\tilde{Y}, \tilde{Z})$ such that $e_Y(- \circ g) = e_Z(\tilde{g} \circ -)$. Then $e_X(- \circ fg) = e_Y(\tilde{f} \circ - \circ g) = e_Z(\tilde{g}\tilde{f} \circ -)$, and hence $\tilde{g}\tilde{f} = \widetilde{gf}$ by uniqueness. ☐

We review the construction of Bondal and Kapranov from the proofs of [BK89, Thm. 2.10, Prop. 3.8], which is explicitly used for the main result of this section.

Theorem 7.3. *Let $\mathcal{T}$ be a triangulated category, linear over a field, with a semiorthogonal decomposition $(\mathcal{U}, \mathcal{V})$. If all contravariant linear cohomological functors $\mathcal{U} \to \mathrm{Vect}$ and $\mathcal{V} \to \mathrm{Vect}$ are representable, then so are all such functors $\mathcal{T} \to \mathrm{Vect}$.*

Proof. Let $h\colon \mathcal{T} \to \mathrm{Vect}$ be a contravariant linear cohomological functor. We proceed in several steps.

(1) By assumption, there are objects $\tilde{U} \in \mathcal{U}$, $\tilde{V}, V' \in \mathcal{V}$, and isomorphisms of functors

$$\eta^{\mathcal{U}} \colon h^{\mathcal{U}}_{\tilde{U}} \cong h|_{\mathcal{U}}, \qquad \eta^{\mathcal{V}} \colon h^{\mathcal{V}}_{\tilde{V}} \cong h|_{\mathcal{V}}, \qquad \text{and} \qquad \theta \colon h^{\mathcal{V}}_{V'} \cong h^{\mathcal{V}}_{\tilde{U}}.$$

Set $e_{\mathcal{U}} := e_{\eta^{\mathcal{U}}}$ and $e_{\mathcal{V}} := e_{\eta^{\mathcal{V}}}$, see Notation 7.1.(b).

(2) Let $u := e_\theta \in h^{V'}_{\tilde{U}}$, and define $v \in h^{V'}_{\tilde{V}}$ by the commutative square

$$\begin{tikzcd}
h^{\tilde{U}}_{\tilde{U}} \arrow[r, dashed] \arrow[d, "\cong"', "\eta^{\mathcal{U}}_{\tilde{U}}"] & h^{V'}_{\tilde{V}} \arrow[d, "\cong"', "\eta^{\mathcal{V}}_{V'}"] & & \mathrm{id}_{\tilde{U}} \arrow[r, mapsto, dashed] \arrow[d, leftrightarrow] & v. \arrow[d, leftrightarrow] \\
h(\tilde{U}) \arrow[r, "h(u)"] & h(V') & & e_{\mathcal{U}} \arrow[r, mapsto] & h(u)(e_{\mathcal{U}}) = h(v)(e_{\mathcal{V}})
\end{tikzcd} \tag{7.1}$$

(3) Complete the morphism defined by u and $-v$ to a distinguished triangle in $\mathcal{T}$ as follows:

$$V' \xrightarrow{\begin{pmatrix} u \\ -v \end{pmatrix}} \tilde{U} \oplus \tilde{V} \xrightarrow{\begin{pmatrix} p & q \end{pmatrix}} \tilde{X} \tag{7.2}$$

It defines a homotopy cartesian square, which fits into a morphism

$$\begin{tikzcd}
V'' \arrow[r] \arrow[d, equal] & V' \arrow[r, "v"] \arrow[d, "u"] \arrow[dr, phantom, "\square"] & \tilde{V} \arrow[d, "q"] \\
V'' \arrow[r, dashed] & \tilde{U} \arrow[r, "p"] & \tilde{X}
\end{tikzcd} \tag{7.3}$$

of distinguished triangles in $\mathcal{T}$, see [Nee01, Lem. 1.4.4]. In the sequel, we establish an isomorphism of functors $h^{\mathcal{T}}_{\tilde{X}} \cong h$.

(4) We have isomorphisms of functors

$$h^{\mathcal{U}}_p \colon h^{\mathcal{U}}_{\tilde{U}} \cong h^{\mathcal{U}}_{\tilde{X}} \quad \text{and} \quad h^{\mathcal{V}}_q \colon h^{\mathcal{V}}_{\tilde{V}} \cong h^{\mathcal{V}}_{\tilde{X}}. \tag{7.4}$$

For $h^{\mathcal{U}}_p$ this is immediate from (7.3) since h^U is homological for all $U \in \mathcal{U}$ and $h^{\mathcal{U}}_{\mathcal{V}} = 0$. For each $V \in \mathcal{V}$ and $f \in h^V_{V'}$ the naturality of θ and the bifunctoriality of $\mathrm{Hom}_{\mathcal{T}}(-,-)$ give rise to a commutative diagram

$$\begin{tikzcd}
h^{V'}_{V'} \arrow[r, "\theta_{V'}", "\cong"'] \arrow[d, "h^f_{V'}"] & h^{V'}_{\tilde{U}} \arrow[d, "h^f_{\tilde{U}}"] & h^{V'}_{V'} \arrow[l, "h^{V'}_u"'] \arrow[d, "h^f_{V'}"] & & \mathrm{id}_{V'} \arrow[r, leftrightarrow] \arrow[d, mapsto] & u \arrow[d, mapsto] & \mathrm{id}_{V'} \arrow[l, mapsto] \arrow[d, mapsto] \\
h^V_{V'} \arrow[r, "\theta_V", "\cong"'] & h^V_{\tilde{U}} & h^V_{V'} \arrow[l, "h^V_u"'] & & f \arrow[r, leftrightarrow] & u \circ f & f, \arrow[l, mapsto]
\end{tikzcd}$$

by which $h^{\mathcal{V}}_u = \theta|_{\mathcal{V}}$ is an isomorphism. Applying the homological functor h^V to (7.3) yields a morphism

$$\begin{tikzcd}
\cdots \arrow[r] & h^V_{V''} \arrow[r] \arrow[d, equal] & h^V_{V'} \arrow[r, "h^V_v"] \arrow[d, "\cong"', "h^V_u"] & h^V_{\tilde V} \arrow[r] \arrow[d, "h^V_q"] & \cdots \\
\cdots \arrow[r] & h^V_{V''} \arrow[r] & h^V_{\tilde U} \arrow[r, "h^V_p"] & h^V_{\tilde X} \arrow[r] & \cdots
\end{tikzcd}$$

of long exact sequences of vector spaces. By the five lemma, $h^{\mathcal{V}}_q$ is an isomorphism as well.

(5) Combining (1) and (4), we obtain the following isomorphisms of functors:

$$\begin{tikzcd}
h^{\mathcal{U}}_{\tilde X} \arrow[rr, dashed, "\rho^{\mathcal{U}}", "\cong"'] & & h|_{\mathcal{U}} & p \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & e_{\mathcal{U}} \arrow[dl, leftrightarrow] \\
& h^{\mathcal{U}}_{\tilde U}, \arrow[ul, "\cong"', "h^{\mathcal{U}}_p"] \arrow[ur, "\cong", "\eta^{\mathcal{U}}"'] & & & \mathrm{id}_{\tilde U}, &
\end{tikzcd} \tag{7.5}$$

$$\begin{tikzcd}
h^{\mathcal{V}}_{\tilde X} \arrow[rr, dashed, "\rho^{\mathcal{V}}", "\cong"'] & & h|_{\mathcal{V}} & q \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & e_{\mathcal{V}} \arrow[dl, leftrightarrow] \\
& h^{\mathcal{V}}_{\tilde V}, \arrow[ul, "\cong"', "h^{\mathcal{U}}_q"] \arrow[ur, "\cong", "\eta^{\mathcal{V}}"'] & & & \mathrm{id}_{\tilde V}\,. &
\end{tikzcd} \tag{7.6}$$

(6) To prepare for the next step, we show that

$$h(u) \circ \rho^{\mathcal{U}}_{\tilde U} = \rho^{\mathcal{V}}_{V'} \circ h^u_{\tilde X}. \tag{7.7}$$

Given $f \in h^{\tilde U}_{\tilde X}$, we use (7.4) to obtain a (unique) $r \in h^{\tilde U}_{\tilde U}$ such that $f = h^{\tilde U}_p(r)$ and a (unique) $s \in h^{V'}_{\tilde V}$ such that $pru = h^u_{\tilde X}(f) = h^{V'}_q(s) = qs \in h^{V'}_{\tilde X}$. Since homotopy cartesian squares are weak pullbacks, we obtain a $w \in h^{V'}_{V'}$ such that

$$\begin{tikzcd}
V' \arrow[r, dashed, "w"] \arrow[d, "u"] \arrow[rr, bend left, "s"] & V' \arrow[r, "v"] \arrow[d, "u"] \arrow[dr, phantom, "\square"] & \tilde V \arrow[d, "q"] \\
\tilde U \arrow[r, "r"] \arrow[rr, bend right, "f"'] & \tilde U \arrow[r, "p"] & \tilde X
\end{tikzcd} \tag{7.8}$$

commutes. This yields

$$\begin{aligned}
h(u)\left(\rho^{\mathcal{U}}_{\tilde U}(f)\right) &= h(u)\left(\rho^{\mathcal{U}}_{\tilde U}\left(h^{\tilde U}_p(r)\right)\right) \overset{(7.5)}{=} h(u)\left(\eta^{\mathcal{U}}_{\tilde U}(r)\right) \underset{(1)}{\overset{7.1.(b)}{=}} h(u)\left(h(r)(e_{\mathcal{U}})\right) = h(ru)(e_{\mathcal{U}}) \\
&\overset{(7.8)}{=} h(uw)(e_{\mathcal{U}}) = h(w)\left(h(u)(e_{\mathcal{U}})\right) \overset{(7.1)}{=} h(w)\left(h(v)(e_{\mathcal{V}})\right) = h(vw)(e_{\mathcal{V}}) \\
&\overset{(7.8)}{=} h(s)(e_{\mathcal{V}}) \underset{(1)}{\overset{7.1.(b)}{=}} \eta^{\mathcal{V}}_{V'}(s) \overset{(7.6)}{=} \rho^{\mathcal{V}}_{V'}\left(h^{V'}_q(s)\right) = \rho^{\mathcal{V}}_{V'}\left(h^u_{\tilde X}(f)\right).
\end{aligned}$$

(7) To obtain a natural transformation $h_{\tilde{X}} \to h$ on all of $\mathcal{T}$, we apply the cohomological functors $h_{\tilde{X}}$ and h to the distinguished triangle (7.2). By (6) and the naturality of $\rho^{\mathcal{V}}\colon h^{\mathcal{V}}_{\tilde{X}} \to h|_{\mathcal{V}}$ applied to $-v\colon V' \to \tilde{V}$, we have a commutative square

$$\begin{tikzcd}
h^{\tilde{U}}_{\tilde{X}} \oplus h^{\tilde{V}}_{\tilde{X}} \arrow[r, "\begin{pmatrix} h^u_{\tilde{X}} & h^{-v}_{\tilde{X}} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \rho^{\mathcal{U}}_{\tilde{U}} & 0 \\ 0 & \rho^{\mathcal{V}}_{\tilde{V}} \end{pmatrix}"', "\cong"] & h^{V'}_{\tilde{X}} \arrow[d, "\rho^{\mathcal{V}}_{V'}"', "\cong"] \\
h(\tilde{U}) \oplus h(\tilde{V}) \arrow[r, "\begin{pmatrix} h(u) & h(-v) \end{pmatrix}"] & h(V').
\end{tikzcd}$$

By Lemma 7.4 for $\mathcal{E} = \mathrm{Vect}$ and the five lemma, we then obtain an isomorphism of long exact sequences of vector spaces

$$\begin{tikzcd}
\cdots \arrow[r] & h^{\tilde{X}}_{\tilde{X}} \arrow[r, "\begin{pmatrix} h^p_{\tilde{X}} \\ h^q_{\tilde{X}} \end{pmatrix}"] \arrow[d, dashed, "\eta_{\tilde{X}}"', "\cong"] & h^{\tilde{U}}_{\tilde{X}} \oplus h^{\tilde{V}}_{\tilde{X}} \arrow[r, "\begin{pmatrix} h^u_{\tilde{X}} & -h^v_{\tilde{X}} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \rho^{\mathcal{U}}_{\tilde{U}} & 0 \\ 0 & \rho^{\mathcal{V}}_{\tilde{V}} \end{pmatrix}"', "\cong"] & h^{V'}_{\tilde{X}} \arrow[r] \arrow[d, "\rho^{\mathcal{V}}_{V'}"', "\cong"] & \cdots \\
\cdots \arrow[r] & h(\tilde{X}) \arrow[r, "\begin{pmatrix} h(p) \\ h(q) \end{pmatrix}"] & h(\tilde{U}) \oplus h(\tilde{V}) \arrow[r, "\begin{pmatrix} h(u) & -h(v) \end{pmatrix}"] & h(V') \arrow[r] & \cdots,
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{id}_{\tilde{X}} \arrow[r, mapsto] \arrow[d, leftrightarrow] & \begin{pmatrix} p \\ q \end{pmatrix} \arrow[d, leftrightarrow] \\
e \arrow[r, mapsto] & \begin{pmatrix} h(p)(e) \\ h(q)(e) \end{pmatrix} = \begin{pmatrix} e_{\mathcal{U}} \\ e_{\mathcal{V}} \end{pmatrix},
\end{tikzcd}$$

(7.9)

which involves a (non-unique) dashed morphism defining an element $e := \eta_{\tilde{X}}(\mathrm{id}_{\tilde{X}}) \in h(\tilde{X})$. This extends to a natural transformation $\eta\colon h_{\tilde{X}} \to h$, see Notation 7.1.(b).

(8) The restrictions $\eta|_{\mathcal{U}}$ and $\eta|_{\mathcal{V}}$ are isomorphisms: Indeed, given $U \in \mathcal{U}$ and $f \in h^U_{\tilde{X}}$, we use (7.4) to obtain a (unique) $\tilde{f} \in h^U_{\tilde{U}}$ such that $f = h^U_p(\tilde{f}) = p\tilde{f}$. This yields

$$\eta_U(f) \overset{7.1.(b)}{=} h(f)(e) = h(\tilde{f})\,(h(p)(e)) \overset{(7.9)}{=} h(\tilde{f})(e_{\mathcal{U}}) \overset{7.1.(b)}{=} \eta^{\mathcal{U}}_U(\tilde{f}) \overset{(7.5)}{=} \rho^{\mathcal{U}}_U\left(h^U_p(\tilde{f})\right) = \rho^{\mathcal{U}}_U(f).$$

Hence, $\eta|_{\mathcal{U}} = \rho^{\mathcal{U}}$ is an isomorphism, see (5). The proof of $\eta|_{\mathcal{V}} = \rho^{\mathcal{V}}$ is analogous.

(9) It remains to be seen that η is an isomorphism. To this end, place an arbitrary object $X \in \mathcal{T}$ into a distinguished triangle $U \longrightarrow X \longrightarrow V$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$, see Definition 2.1.(b). Applying the natural transformation η yields a morphism of long exact sequences of vector spaces

$$\begin{tikzcd}
\cdots \arrow[r] & h_{\tilde{X}}(V) \arrow[r] \arrow[d, "\cong"', "\eta_V"] & h_{\tilde{X}}(X) \arrow[r] \arrow[d, "\eta_X"] & h_{\tilde{X}}(U) \arrow[r] \arrow[d, "\cong"', "\eta_U"] & \cdots \\
\cdots \arrow[r] & h(V) \arrow[r] & h(X) \arrow[r] & h(U) \arrow[r] & \cdots,
\end{tikzcd}$$

with isomorphisms as indicated due to (8). By the five lemma, η_X is an isomorphism. $\square$

Lemma 7.4. *Let $\mathcal{E}$ be an exact category with* $\mathrm{Proj}(\mathcal{E}) = \mathcal{E}$*, or, equivalently,* $\mathrm{Inj}(\mathcal{E}) = \mathcal{E}$*. Then any solid commutative diagram over $\mathcal{E}$ with exact rows as below can be completed by a (non-unique) dashed morphism:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed] & D \arrow[r] \arrow[d] & E \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

$\square$

With an additional hypothesis, a modification of the proof of Theorem 7.3 yields a refined result:

Theorem 7.5. *Let $\mathcal{T}$ be a triangulated category, linear over a field, with semiorthogonal decompositions $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{V}^{\perp})$. A contravariant linear cohomological functor $h\colon \mathcal{T} \to$* Vect *is representable if $h|_{\mathcal{U}}$ and $h|_{\mathcal{V}}$ are so.*

Proof. We only describe the changes to the proof of Theorem 7.3 to eliminate θ in step (1). We pick a distinguished triangle $V' \xrightarrow{u} \tilde{U} \longrightarrow V^{\perp}$ with $V' \in \mathcal{V}$ and $V^{\perp} \in \mathcal{V}^{\perp}$. In step (2), this u is used instead of e_θ to define v. In step (3), we extend the diagram (7.3) to

$$\begin{tikzcd}
V'' \arrow[r] \arrow[d, equal] & V' \arrow[r, "v"] \arrow[d, "u"] \arrow[dr, phantom, "\square"] & \tilde{V} \arrow[d, "q"] \\
V'' \arrow[r] & \tilde{U} \arrow[r, "p"] \arrow[d] & \tilde{X} \arrow[d, dashed] \\
 & V^{\perp} \arrow[r, equal] & V^{\perp}
\end{tikzcd} \tag{7.10}$$

with columns distinguished triangles, see [Nee01, Lem. 1.4.4]. In step (4), the argument for $h_p^{\mathcal{U}}$ then also applies to $h_q^{\mathcal{V}}$. From step (5) onward, the proof remains unchanged. $\square$

The previous construction is now applied to dualized hom-functors, see [BK89, Prop. 3.8]:

Corollary 7.6. *Let $\mathcal{T}$ be a triangulated category, linear over a field, with semiorthogonal decompositions $({}^{\perp}\mathcal{U}, \mathcal{U})$, $(\mathcal{U}, \mathcal{V})$, and $(\mathcal{V}, \mathcal{V}^{\perp})$. Given an object $X \in \mathcal{T}$, consider distinguished triangles ${}^{\perp}U \to X \xrightarrow{t_U} U$ and $U' \to X \xrightarrow{t_V} V$ with ${}^{\perp}U \in {}^{\perp}\mathcal{U}$, $U, U' \in \mathcal{U}$, and $V \in \mathcal{V}$. Then the functor $\left(h_{\mathcal{T}}^X\right)^*$ is representable if the functors $\left(h_{\mathcal{U}}^U\right)^*$ and $\left(h_{\mathcal{V}}^V\right)^*$ are so.*

Proof. We apply Theorem 7.5 to $h := \left(h_{\mathcal{T}}^X\right)^*$. Note that $h_{\mathcal{U}}^{t_U}\colon h_{\mathcal{U}}^U \cong h_{\mathcal{U}}^X$ and $h_{\mathcal{V}}^{t_V}\colon h_{\mathcal{V}}^V \cong h_{\mathcal{V}}^X$ are isomorphisms of functors. By hypothesis, there are $\tilde{U} \in \mathcal{U}$, $\tilde{V} \in \mathcal{V}$, and isomorphisms of functors $\eta_U\colon h_{\tilde{U}}^{\mathcal{U}} \cong \left(h_{\mathcal{U}}^U\right)^*$ and $\eta_V\colon h_{\tilde{V}}^{\mathcal{V}} \cong \left(h_{\mathcal{V}}^V\right)^*$. By composition, we obtain isomorphisms of functors

$$\begin{tikzcd}
 & \left(h^X_{\mathcal{U}}\right)^* = h|_{\mathcal{U}} \arrow[d, "\cong"', "\left(h^{t_U}_{\mathcal{U}}\right)^*"] & h|_{\mathcal{V}} = \left(h^X_{\mathcal{V}}\right)^* \arrow[d, "\left(h^{t_V}_{\mathcal{V}}\right)^*"', "\cong"] & \\
h^{\mathcal{U}}_{\tilde{U}} \arrow[r, "\cong", "\eta_U"'] \arrow[ur, dashed, "\eta^{\mathcal{U}}", "\cong"'] & \left(h^U_{\mathcal{U}}\right)^* & \left(h^V_{\mathcal{V}}\right)^* & h^{\mathcal{V}}_{\tilde{V}}, \arrow[l, "\cong"', "\eta_V"] \arrow[ul, dashed, "\eta^{\mathcal{V}}"', "\cong"]
\end{tikzcd} \tag{7.11}$$

$$\begin{tikzcd}
 & e_{\mathcal{U}} \arrow[d, leftrightarrow] \arrow[dl, dashed, leftrightarrow] & e_{\mathcal{V}} \arrow[d, leftrightarrow] \arrow[dr, dashed, leftrightarrow] & \\
\mathrm{id}_{\tilde{U}} \arrow[r, leftrightarrow] & e_U = e_{\mathcal{U}}(- \circ t_U) & e_{\mathcal{V}}(- \circ t_V) = e_V \arrow[r, leftrightarrow] & \mathrm{id}_{\tilde{V}} .
\end{tikzcd}$$

This shows that $h|_{\mathcal{U}}$ and $h|_{\mathcal{V}}$ are representable and the claim follows from Theorem 7.5. □

We apply the preceeding construction to prove Theorem E from the introduction:

Theorem 7.7. *Let $\mathcal{F}$ be a Frobenius category, linear over field, $l \in \mathbb{N}$, and $(X, \alpha) \in \mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$. Consider two diagrams*

$$\begin{tikzcd}
X^{0,0} \arrow[r, tail, "\alpha^{0,0}"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & X^{0,1} \arrow[r, tail, "\alpha^{0,1}"] \arrow[d, two heads, "\beta^{0,1}"] \arrow[dr, phantom, "\square"] & X^{0,2} \arrow[r, tail, "\alpha^{0,2}"] \arrow[d, two heads, "\beta^{0,2}"] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail, "\alpha^{0,l-3}"] \arrow[dr, phantom, "\square"] & X^{0,l-2} \arrow[r, tail, "\alpha^{0,l-2}"] \arrow[d, two heads, "\beta^{0,l-2}"] \arrow[dr, phantom, "\square"] & X^{0,l-1} \arrow[r, tail, "\alpha^{0,l-1}"] \arrow[d, two heads, "\beta^{0,l-1}"] \arrow[dr, phantom, "\square"] & X^{0,l} \arrow[d, two heads, "\beta^{0,l}"] \\
0 \arrow[r, tail] & X^{1,1} \arrow[r, tail, "\alpha^{1,1}"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & X^{1,2} \arrow[r, tail, "\alpha^{1,2}"] \arrow[d, two heads, "\beta^{1,2}"] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail, "\alpha^{1,l-3}"] \arrow[dr, phantom, "\square"] & X^{1,l-2} \arrow[r, tail, "\alpha^{1,l-2}"] \arrow[d, two heads, "\beta^{1,l-2}"] \arrow[dr, phantom, "\square"] & X^{1,l-1} \arrow[r, tail, "\alpha^{1,l-1}"] \arrow[d, two heads, "\beta^{1,l-1}"] \arrow[dr, phantom, "\square"] & X^{1,l} \arrow[d, two heads, "\beta^{1,l}"] \\
 & 0 \arrow[r, tail] & X^{2,2} \arrow[r, tail, "\alpha^{2,2}"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail, "\alpha^{2,l-3}"] \arrow[dr, phantom, "\square"] & X^{2,l-2} \arrow[r, tail, "\alpha^{2,l-2}"] \arrow[d, two heads, "\beta^{2,l-2}"] \arrow[dr, phantom, "\square"] & X^{2,l-1} \arrow[r, tail, "\alpha^{2,l-1}"] \arrow[d, two heads, "\beta^{2,l-1}"] \arrow[dr, phantom, "\square"] & X^{2,l} \arrow[d, two heads, "\beta^{2,l}"] \\
 & & 0 \arrow[r, tail] & \cdots & \vdots \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \vdots \arrow[d, two heads, "\beta^{l-2,l-1}"] \arrow[dr, phantom, "\square"] & \vdots \arrow[d, two heads, "\beta^{l-2,l}"] \\
 & & & & 0 \arrow[r, tail] & X^{l-1,l-1} \arrow[r, tail, "\alpha^{l-1,l-1}"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & X^{l-1,l} \arrow[d, two heads, "\beta^{l-1,l}"] \\
 & & & & & 0 \arrow[r, tail] & X^{l,l},
\end{tikzcd} \tag{7.12}$$

$$\begin{tikzcd}
\tilde{X}^{0,0} \arrow[r, "\tilde{\beta}^{0,0}"] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \tilde{X}^{0,1} \arrow[r, "\tilde{\beta}^{0,1}"] \arrow[d, tail, "\tilde{\alpha}^{0,1}"] \arrow[dr, phantom, "\square"] & \tilde{X}^{0,2} \arrow[r, "\tilde{\beta}^{0,2}"] \arrow[d, tail, "\tilde{\alpha}^{0,2}"] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, "\tilde{\beta}^{0,l-3}"] \arrow[dr, phantom, "\square"] & \tilde{X}^{0,l-2} \arrow[r, "\tilde{\beta}^{0,l-2}"] \arrow[d, tail, "\tilde{\alpha}^{0,l-2}"] \arrow[dr, phantom, "\square"] & \tilde{X}^{0,l-1} \arrow[r, "\tilde{\beta}^{0,l-1}"] \arrow[d, tail, "\tilde{\alpha}^{0,l-1}"] \arrow[dr, phantom, "\square"] & \tilde{X}^{0,l} \arrow[d, tail, "\tilde{\alpha}^{0,l}"] \\
I^0 \arrow[r] & \tilde{X}^{1,1} \arrow[r, "\tilde{\beta}^{1,1}"] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \tilde{X}^{1,2} \arrow[r, "\tilde{\beta}^{1,2}"] \arrow[d, tail, "\tilde{\alpha}^{1,2}"] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, "\tilde{\beta}^{1,l-3}"] \arrow[dr, phantom, "\square"] & \tilde{X}^{1,l-2} \arrow[r, "\tilde{\beta}^{1,l-2}"] \arrow[d, tail, "\tilde{\alpha}^{1,l-2}"] \arrow[dr, phantom, "\square"] & \tilde{X}^{1,l-1} \arrow[r, "\tilde{\beta}^{1,l-1}"] \arrow[d, tail, "\tilde{\alpha}^{1,l-1}"] \arrow[dr, phantom, "\square"] & \tilde{X}^{1,l} \arrow[d, tail, "\tilde{\alpha}^{1,l}"] \\
 & I^1 \arrow[r] & \tilde{X}^{2,2} \arrow[r, "\tilde{\beta}^{2,2}"] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, "\tilde{\beta}^{2,l-3}"] & \tilde{X}^{2,l-2} \arrow[r, "\tilde{\beta}^{2,l-2}"] \arrow[d, tail, "\tilde{\alpha}^{2,l-2}"] \arrow[dr, phantom, "\square"] & \tilde{X}^{2,l-1} \arrow[r, "\tilde{\beta}^{2,l-1}"] \arrow[d, tail, "\tilde{\alpha}^{2,l-1}"] \arrow[dr, phantom, "\square"] & \tilde{X}^{2,l} \arrow[d, tail, "\tilde{\alpha}^{2,l}"] \\
 & & I^2 \arrow[r] & \cdots & \vdots \arrow[d, tail] \arrow[dr, phantom, "\square"] & \vdots \arrow[d, tail, "\tilde{\alpha}^{l-2,l-1}"] \arrow[dr, phantom, "\square"] & \vdots \arrow[d, tail, "\tilde{\alpha}^{l-2,l}"] \\
 & & & & I^{l-2} \arrow[r] & \tilde{X}^{l-1,l-1} \arrow[r, "\tilde{\beta}^{l-1,l-1}"] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \tilde{X}^{l-1,l} \arrow[d, tail, "\tilde{\alpha}^{l-1,l}"] \\
 & & & & & I^{l-1} \arrow[r] & \tilde{X}^{l,l}
\end{tikzcd} \tag{7.13}$$

of bicartesian squares in $\mathcal{F}$*, where* $(X^{0,\bullet}, \alpha^{0,\bullet}) = (X, \alpha)$ *and* $I^k \in \operatorname{Inj}(\mathcal{F})$ *for* $k \in \{0, \ldots, l-1\}$*. Set* $(\tilde{X}, \tilde{\alpha}) := (\tilde{X}^{\bullet,l}, \tilde{\alpha}^{\bullet,l})$*. Suppose that there are isomorphisms of functors*

$$\eta^{i,j} := \eta_{X^{i,j}} \colon h^{\mathcal{M}_0}_{\tilde{X}^{i,j}} \cong \left(h^{X^{i,j}}_{\mathcal{M}_0}\right)^*,$$

where $0 \leq i \leq j \leq l$*, such that* $e_{X^{i,j}}\left(- \circ \overline{\beta^{i,j}}\right) = e_{X^{i+1,j}}\left(\overline{\tilde{\alpha}^{i,j}} \circ -\right)$ *on* $\operatorname{Hom}_{\mathcal{M}_0}(X^{i+1,j}, \tilde{X}^{i,j})$ *for all* $0 \leq i < j \leq l$*, see Notation 7.1.(c). Then there exists an isomorphism of functors*

$$\eta_X \colon h^{\mathcal{M}_l}_{\tilde{X}} \cong \left(h^X_{\mathcal{M}_l}\right)^*$$

such that $e_X\left(\overline{(0, \ldots, 0, \psi\beta^{l-1,l} \cdots \beta^{0,l})}\right) = e_{X^{l,l}}\left(\overline{\psi}\right)$ *for all* $\psi \in \operatorname{Hom}_{\mathcal{F}}(X^{l,l}, \tilde{X}^l)$*.*

Proof. The claim is trivial for $l = 0$. For $l > 0$, we proceed as in Theorem 7.5 and Corollary 7.6 with the notation used there. To this end, we set

$$h := \left(h^X_{\mathcal{M}_l}\right)^*, \ {}^{\perp}\mathcal{U} := \Gamma^{l-1}, \ \mathcal{U} := \Delta^{[l-1,l]}, \ \mathcal{V} := \Gamma^l, \text{ and } \mathcal{V}^{\perp} := \Gamma^{[0,l-1]}, \tag{7.14}$$

see Propositions 4.1 and 4.4. Contracting the $(l-1)$st column and the lth row of both (7.12) and (7.13) establishes the given setup for $\gamma^{l-1}(X) \in \mathcal{M}_{l-1}$ and $\widetilde{\gamma^{l-1}(X)} = \gamma^l(\tilde{X})$, see Lemma 7.2. By induction, we obtain an isomorphism of functors $\eta_{\gamma^{l-1}(X)} \colon h^{\mathcal{M}_{l-1}}_{\gamma^l(\tilde{X})} \xrightarrow{\cong} \left(h^{\gamma^{l-1}(X)}_{\mathcal{M}_{l-1}}\right)^*$ such that

$$e_{\gamma^{l-1}(X)}\left(\overline{(0, \ldots, 0, \psi\beta^{l-2,l} \cdots \beta^{0,l})}\right) = e_{X^{l-1,l}}\left(\overline{\psi}\right) \tag{7.15}$$

for all $\psi \in \operatorname{Hom}_{\mathcal{F}}(X^{l-1,l}, \tilde{X}^{l-1})$. To process Corollary 7.6, consider the distinguished triangles

$$\begin{tikzcd}
{}^{\perp}U \arrow[d] & 0 \arrow[r, equal] \arrow[d] & \cdots \arrow[r, equal] & 0 \arrow[r, tail] \arrow[d] & Y^{l-1} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & P(X^l) \arrow[d, two heads, "p_{X^l}"] \\
X \arrow[d, "t_U"] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{l-3}"] & X^{l-2} \arrow[r, tail, "\alpha^{l-2}"] \arrow[d, equal] & X^{l-1} \arrow[r, tail, "\alpha^{l-1}"] \arrow[d, tail, "\alpha^{l-1}"] & X^l \arrow[d, equal] \\
U & X^0 \arrow[r, tail, "\alpha^0"] & \cdots \arrow[r, tail, "\alpha^{l-3}"] & X^{l-2} \arrow[r, tail, "\alpha^{l-1}\alpha^{l-2}"] & X^l \arrow[r, equal] & X^l,
\end{tikzcd} \tag{7.16}$$

$$\begin{tikzcd}
U' \arrow[d] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d, equal] & \cdots \arrow[r, tail, "\alpha^{l-3}"] & X^{l-2} \arrow[r, tail, "\alpha^{l-2}"] \arrow[d, equal] & X^{l-1} \arrow[r, equal] \arrow[d, equal] & X^{l-1} \arrow[d, tail, "\alpha^{l-1}"] \\
X \arrow[d, "t_V"] & X^0 \arrow[r, tail, "\alpha^0"] \arrow[d] & \cdots \arrow[r, tail, "\alpha^{l-3}"] & X^{l-2} \arrow[r, tail, "\alpha^{l-2}"] \arrow[d] & X^{l-1} \arrow[r, tail, "\alpha^{l-1}"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & X^l \arrow[d, two heads, "\beta^{l-1,l}\cdots\beta^{0,l}"] \\
V & 0 \arrow[r, equal] & \cdots \arrow[r, equal] & 0 \arrow[r, equal] & 0 \arrow[r, tail] & X^{l,l}
\end{tikzcd} \tag{7.17}$$

in $\mathcal{M}_l$, see Propositions 4.4.(a).(i) and 4.4.(b).(i).

To represent $\left(h^U_{\mathcal{U}}\right)^*$ and hence $h|_{\mathcal{U}}$, we use the equivalence $\mathcal{U} \xrightarrow{\simeq} \mathcal{M}_{l-1}$, given by the restriction of $\underline{\gamma}^{l-1}$, see Lemma 3.5, and (7.11): Set

$$\tilde{U}\colon\ \tilde{X}^0 \xrightarrow{\tilde{\alpha}^0} \tilde{X}^1 \xrightarrow{\tilde{\alpha}^1} \dots \xrightarrow{\tilde{\alpha}^{l-2}} \tilde{X}^{l-1} = \tilde{X}^{l-1}.$$

Since $\gamma^{l-1}(U) = \gamma^{l-1}(X)$ and $\gamma^{l-1}(\tilde{U}) = \gamma^l(\tilde{X})$, we obtain a commutative diagram

$$\begin{tikzcd}
 & h|_{\mathcal{U}} \arrow[d, "\cong"', "\left(h^{t_U}_{\mathcal{U}}\right)^*"] & & e_{\mathcal{U}} \arrow[dll, leftrightarrow] \arrow[d, leftrightarrow] \\
h^{\mathcal{U}}_{\tilde{U}} \arrow[ur, dashed, "\eta^{\mathcal{U}}", "\cong"'] \arrow[r, dashed, "\cong", "\eta_U"'] \arrow[d, "\underline{\gamma}^{l-1}"', "\cong"] & \left(h^U_{\mathcal{U}}\right)^* & \mathrm{id}_{\tilde{U}} \arrow[r, leftrightarrow] \arrow[dd, leftrightarrow] & e_{\mathcal{U}}(- \circ t_U) = e_{\gamma^{l-1}(X)} \circ \underline{\gamma}^{l-1} \arrow[dd, leftrightarrow] \\
h^{\underline{\gamma}^{l-1}(\mathcal{U})}_{\gamma^{l-1}(\tilde{U})} \arrow[r, dashed, "\cong"'] \arrow[d, equal] & \left(h^{\gamma^{l-1}(X)}_{\underline{\gamma}^{l-1}(\mathcal{U})}\right)^* \arrow[u, "\cong", "(\underline{\gamma}^{l-1})^*"'] \arrow[d, equal] & & \\
h^{\mathcal{M}_{l-1}}_{\gamma^l(\tilde{X})} \circ \underline{\gamma}^{l-1} \arrow[r, "\eta_{\gamma^{l-1}(X)} \circ \underline{\gamma}^{l-1}", "\cong"'] & \left(h^{\gamma^{l-1}(X)}_{\mathcal{M}_{l-1}}\right)^* \circ \underline{\gamma}^{l-1}, & \mathrm{id}_{\gamma^l(\tilde{X})} \arrow[r, leftrightarrow] & e_{\gamma^{l-1}(X)},
\end{tikzcd} \tag{7.18}$$

of isomorphisms of functors. To represent $\left(h^V_{\mathcal{V}}\right)^*$ and hence $h|_{\mathcal{V}}$, we use the equivalence $\mathcal{V} \xrightarrow{\simeq} \mathcal{M}_0$, given by the restriction of $\underline{\gamma}^{l^c}$, see Construction 5.2, and (7.11): Set

$$\tilde{V}\colon\ 0 = 0 = \dots = 0 \rightarrowtail \tilde{X}^l.$$

Since $\gamma^{l^c}(V) = X^{l,l}$ and $\gamma^{l^c}(\tilde{V}) = \tilde{X}^l$, we obtain a commutative diagram

$$\begin{tikzcd}
h|_{\mathcal{V}} \arrow[d, "\left(h^{t_V}_{\mathcal{U}}\right)^*"', "\cong"] & \\
\left(h^V_{\mathcal{V}}\right)^* & h^{\mathcal{V}}_{\tilde{V}} \arrow[l, dashed, "\cong", "\eta_V"'] \arrow[ul, dashed, "\eta^{\mathcal{V}}"', "\cong"] \arrow[d, "\cong"', "\underline{\gamma}^{l^c}"] \\
\left(h^{\gamma^{l^c}(V)}_{\underline{\gamma}^{l^c}(\mathcal{V})}\right)^* \arrow[u, "\left(\underline{\gamma}^{l^c}\right)^*", "\cong"'] \arrow[d, equal] & h^{\underline{\gamma}^{l^c}(\mathcal{V})}_{\gamma^{l^c}(\tilde{V})} \arrow[l, dashed, "\cong"'] \arrow[d, equal] \\
\left(h^{X^{l,l}}_{\mathcal{M}_0}\right)^* \circ \underline{\gamma}^{l^c} & h^{\mathcal{M}_0}_{\tilde{X}^l} \circ \underline{\gamma}^{l^c}, \arrow[l, "\eta^{l,l} \circ \underline{\gamma}^{l^c}"', "\cong"]
\end{tikzcd}
\qquad
\begin{tikzcd}
e_{\mathcal{V}} \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & \\
e_{X^{l,l}} \circ \underline{\gamma}^{l^c} = e_{\mathcal{V}}(- \circ t_V) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathrm{id}_{\tilde{V}} \arrow[d, leftrightarrow] \\
e_{X^{l,l}} \arrow[r, leftrightarrow] & \mathrm{id}_{\tilde{X}^l},
\end{tikzcd}
\tag{7.19}$$

of isomorphisms of functors. As in the proof of Theorem 7.5, we consider the distinguished triangle

$$\begin{tikzcd}
V' \arrow[d, "u"] & 0 \arrow[r, equal] \arrow[d] & \cdots \arrow[r, equal] & 0 \arrow[r, equal] \arrow[d] & 0 \arrow[r, tail] \arrow[d] & \tilde{X}^{l-1} \arrow[d, equal] \\
\tilde{U} \arrow[d] & \tilde{X}^0 \arrow[r, tail, "\tilde{\alpha}^0"] \arrow[d, equal] & \cdots \arrow[r, tail, "\tilde{\alpha}^{l-3}"] & \tilde{X}^{l-2} \arrow[r, tail, "\tilde{\alpha}^{l-2}"] \arrow[d, equal] & \tilde{X}^{l-1} \arrow[r, equal] \arrow[d, equal] & \tilde{X}^{l-1} \arrow[d, tail, "i_{\tilde{X}^{l-1}}"] \\
V^{\perp} & \tilde{X}^0 \arrow[r, tail, "\tilde{\alpha}^0"] & \cdots \arrow[r, tail, "\tilde{\alpha}^{l-3}"] & \tilde{X}^{l-2} \arrow[r, tail, "\tilde{\alpha}^{l-2}"] & \tilde{X}^{l-1} \arrow[r, tail] & I(\tilde{X}^{l-1}),
\end{tikzcd}
\tag{7.20}$$

in $\mathcal{M}_l$, see Proposition 4.1.(a).

Finally, we show that $\tilde{X}$ results from the construction in Theorem 7.3: To this end, we compute $v \in \mathrm{Hom}_{\mathcal{M}_l}(V', \tilde{V})$ by combining (7.18) and (7.19), see (7.1):

$$\begin{tikzcd}
 & h(\tilde{U}) \arrow[r, "h(u)"] \arrow[d, "\cong"', "\left(h^{t_U}_{\tilde{U}}\right)^*"] & h(V') \arrow[d, "\left(h^{t_V}_{V'}\right)^*"', "\cong"] & \\
h^{\tilde{U}}_{\tilde{U}} \arrow[ur, "\eta^{\mathcal{U}}_{\tilde{U}}", "\cong"'] \arrow[r, "\cong"] \arrow[d, "\underline{\gamma}^{l-1}"', "\cong"] & \left(h^U_{\tilde{U}}\right)^* \arrow[r, dashed] & \left(h^V_{V'}\right)^* & h^{V'}_{\tilde{V}} \arrow[l, "\cong"'] \arrow[ul, "\eta^{\mathcal{V}}_{V'}"', "\cong"] \arrow[d, "\cong"', "\underline{\gamma}^{l^c}"] \\
h^{\gamma^l(\tilde{X})}_{\gamma^l(\tilde{X})} \arrow[r, "\left(\eta_{\gamma^{l-1}(X)} \circ \underline{\gamma}^{l-1}\right)_{\gamma^l(\tilde{X})}", "\cong"'] & \left(h^{\gamma^{l-1}(X)}_{\gamma^l(\tilde{X})}\right)^* \arrow[u, "\cong", "\left(\underline{\gamma}^{l-1}\right)^*"'] \arrow[r, dashed] & \left(h^{X^{l,l}}_{\tilde{X}^{l-1}}\right)^* \arrow[u, "\left(\underline{\gamma}^{l^c}\right)^*", "\cong"'] & h^{\tilde{X}^{l-1}}_{\tilde{X}^l} \arrow[l, "\left(\eta^{l,l} \circ \underline{\gamma}^{l^c}\right)_{\tilde{X}^{l-1}}"', "\cong"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & e_{\gamma^{l-1}(X)} \circ \underline{\gamma}^{l-1} \arrow[r, mapsto] \arrow[d, leftrightarrow, "(1)"] & \left(e_{\gamma^{l-1}(X)} \circ \underline{\gamma}^{l-1}\right)(u \circ -) \arrow[d, leftrightarrow] & \\
\mathrm{id}_{\tilde U} \arrow[ru, dashed] \arrow[r, dashed, leftrightarrow] \arrow[d, leftrightarrow] & e_{\gamma^{l-1}(X)} \circ \underline{\gamma}^{l-1} \arrow[r, dashed, mapsto] \arrow[d, leftrightarrow] & \left(e_{\gamma^{l-1}(X)} \circ \underline{\gamma}^{l-1}\right)(u \circ - \circ t_V) \arrow[r, dashed, leftrightarrow] \arrow[d, leftrightarrow, "(2)"'] & \underline{\delta}^{l^c}\,\overline{\tilde\alpha^{l-1}} = v \arrow[lu, dashed, leftrightarrow] \arrow[d, leftrightarrow] \\
\mathrm{id}_{\gamma^l(\tilde X)} \arrow[r, leftrightarrow] & e_{\gamma^{l-1}(X)} \arrow[r, dashed, mapsto] & e_{X^{l,l}}\left(\overline{\tilde\alpha^{l-1}} \circ -\right) \arrow[r, leftrightarrow] & \overline{\tilde\alpha^{l-1}}
\end{tikzcd}$$

The correspondence (1) holds since $\underline{\gamma}^{l-1}(t_U) = \mathrm{id}_{\gamma^{l-1}(X)}$, see (7.16). To see (2), note that a general element of $h^V_{V'}$ is of the form $\overline{(0,\dots,0,\psi)}$, where $\psi \in \mathrm{Hom}_{\mathcal{F}}(X^{l,l},\tilde X^{l-1})$. Using the hypothesis $e_{X^{l-1,l}}\left(- \circ \overline{\beta^{l-1,l}}\right) = e_{X^{l,l}}\left(\overline{\tilde\alpha^{l-1,l}} \circ -\right)$, we compute

$$\begin{aligned}
e_{X^{l,l}}\left(\overline{\tilde\alpha^{l-1}} \circ \underline{\gamma}^{l^c}\left(\overline{(0,\dots,0,\psi)}\right)\right) &= e_{X^{l,l}}\left(\overline{\tilde\alpha^{l-1}\psi}\right) = e_{X^{l-1,l}}\left(\overline{\psi\beta^{l-1,l}}\right) \\
&\overset{(7.15)}{=} e_{\gamma^{l-1}(X)}\left(\overline{(0,\dots,0,\psi\beta^{l-1,l}\beta^{l-2,l}\cdots\beta^{0,l})}\right) \\
&\underset{(7.20)}{\overset{(7.17)}{=}} \left(e_{\gamma^{l-1}(X)} \circ \underline{\gamma}^{l-1}\right)\left(u \circ \overline{(0,\dots,0,\psi)} \circ t_V\right).
\end{aligned}$$

All other correspondences are obvious or due to the diagram's commutativity.

We put u, v, and $\tilde X$ in a commutative square

$$\begin{tikzcd}
 & \tilde V \arrow[dd, "q"] & & 0 \arrow[rr, equal] \arrow[dd] & & \cdots \arrow[rr, equal] & & 0 \arrow[rr, tail] \arrow[dd] & & \tilde X^l \arrow[dd, equal] \\
V' \arrow[ru, tail, "v"] \arrow[dd, "u"] & & 0 \arrow[ru, equal] \arrow[rr, equal] \arrow[dd] & & \cdots \arrow[rr, equal] & & 0 \arrow[ru, equal] \arrow[rr, tail] \arrow[dd] & & \tilde X^{l-1} \arrow[ru, tail, "\tilde\alpha^{l-1}"] \arrow[dd, equal] & \\
 & \tilde X & & \tilde X^0 \arrow[rr, tail, "\tilde\alpha^0"] & & \cdots \arrow[rr, tail, "\tilde\alpha^{l-2}"] & & \tilde X^{l-1} \arrow[rr, tail, "\tilde\alpha^{l-1}"] & & \tilde X^l \\
\tilde U \arrow[ru, tail, "p"'] & & \tilde X^0 \arrow[ru, equal] \arrow[rr, tail, "\tilde\alpha^0"] & & \cdots \arrow[rr, tail, "\tilde\alpha^{l-2}"] & & \tilde X^{l-1} \arrow[ru, equal] \arrow[rr, equal] & & \tilde X^{l-1} \arrow[ru, tail, "\tilde\alpha^{l-1}"'] &
\end{tikzcd}$$

in $\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$. It yields a (termwise) short exact sequence $V' \overset{\begin{pmatrix} u \\ -v \end{pmatrix}}{\rightarrowtail} \tilde U \oplus \tilde V \overset{\begin{pmatrix} p & q \end{pmatrix}}{\twoheadrightarrow} \tilde X$ in $\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$, where $p = (\mathrm{id}_{\tilde X^0},\dots,\mathrm{id}_{\tilde X^{l-1}},\tilde\alpha^{l-1})$ and $q = (0,\dots,0,\mathrm{id}_{\tilde X^l})$. The corresponding distinguished triangle, see Lemma 2.17, matches (7.2). Therefore, Theorem 7.3 yields an isomorphism of functors $\eta_X := \eta\colon h^{\mathcal{M}_l}_{\tilde X} \cong h$ defined by $e_X := e \in h(\tilde X)$, see step (7) of the proof.

It remains to verify that $e_X\left(\overline{(0,\ldots,0,\psi\beta^{l-1,l}\cdots\beta^{0,l})}\right) = e_{X^{l,l}}\left(\overline{\psi}\right)$ for all $\psi \in \operatorname{Hom}_{\mathcal{F}}(X^{l,l},\tilde{X}^l)$:

$$\begin{aligned} e_X\left(\overline{(0,\ldots,0,\psi\beta^{l-1,l}\cdots\beta^{0,l})}\right) &= e\left(q \circ \overline{(0,\ldots,0,\psi\beta^{l-1,l}\cdots\beta^{0,l})}\right) \\ &\overset{(7.14)}{=} h(q)(e)\left(\overline{(0,\ldots,0,\psi\beta^{l-1,l}\cdots\beta^{0,l})}\right) \\ &\overset{(7.9)}{=} e_{\mathcal{V}}\left(\overline{(0,\ldots,0,\psi\beta^{l-1,l}\cdots\beta^{0,l})}\right) \overset{(7.17)}{=} e_{\mathcal{V}}\left(\overline{(0,\ldots,0,\psi)} \circ t_V\right) \\ &\overset{(7.19)}{=} \left(e_{X^{l,l}} \circ \underline{\gamma}^{l^c}\right)\left(\overline{(0,\ldots,0,\psi)}\right) = e_{X^{l,l}}\left(\overline{\psi}\right) \end{aligned}$$ □

It remains to construct a diagram of the form (7.13), suitable for Theorem 7.7:

Proposition 7.8. *Let $\mathcal{F}$ be a Frobenius category, linear over a field, such that $\underline{\mathcal{F}}$ is hom-finite. Given any diagram in $\mathcal{F}$ of the form* (7.12)*, suppose that any object $X \in \mathcal{F}$ in the diagram admits an isomorphism of functors $\eta_X \colon h_{\tilde{X}} \cong \left(h^X\right)^*$ of contravariant functors $\underline{\mathcal{F}} \to$* Vect*, where $\tilde{X} \in \mathcal{F}$. Then any diagram in $\mathcal{F}$ of the form* (7.13) *with $e_{X^{0,j}}\left(- \circ \overline{\alpha^{0,j}}\right) = e_{X^{0,j+1}}\left(\overline{\tilde{\beta}^{0,j}} \circ -\right)$ on* $\operatorname{Hom}_{\underline{\mathcal{F}}}(X^{0,j+1},\tilde{X}^{0,j})$ *for all $j \in \{0,\ldots,l-1\}$ meets the requirements stated in Theorem 7.7.*

Proof. Using the notation from (7.12), for each $i, j \in \{0,\ldots,l\}$ with $i \leq j$, there is an isomorphism of functors $\hat{\eta}_{X^{i,j}} \colon h_{\hat{X}^{i,j}} \cong \left(h^{X^{i,j}}\right)^*$ defining $\hat{e}_{X^{i,j}} := e_{\hat{\eta}_{X^{i,j}}} \in \left(h^{X^{i,j}}_{\hat{X}^{i,j}}\right)^*$, where $\hat{X}^{i,j} \in \mathcal{F}$, and morphisms $\hat{\alpha}^{i,j} \colon \hat{X}^{i,j} \to \hat{X}^{i+1,j}$ and $\hat{\beta}^{i,j} \colon \hat{X}^{i,j} \to \hat{X}^{i,j+1}$ in $\mathcal{F}$ such that

- $\hat{e}_{X^{i,j}}\left(- \circ \overline{\alpha^{i,j}}\right) = \hat{e}_{X^{i,j+1}}\left(\overline{\hat{\beta}^{i,j}} \circ -\right)$ on $\operatorname{Hom}_{\underline{\mathcal{F}}}(X^{i,j+1},\hat{X}^{i,j})$ for all $0 \leq i \leq j < l$,
- $\hat{e}_{X^{i,j}}\left(- \circ \overline{\beta^{i,j}}\right) = \hat{e}_{X^{i+1,j}}\left(\overline{\hat{\alpha}^{i,j}} \circ -\right)$ on $\operatorname{Hom}_{\underline{\mathcal{F}}}(X^{i+1,j},\hat{X}^{i,j})$ for all $0 \leq i < j \leq l$.

For $j \in \{0,\ldots,l\}$, set $\tilde{X}^{0,j} := \hat{X}^{0,j}$ and $\tilde{\beta}^{0,j} := \hat{\beta}^{0,j}$ if $j < l$ to construct (7.13) by pushouts and a choice of admissible monics $\tilde{\alpha}^{j,j} \colon \tilde{X}^{j,j} \rightarrowtail I^j =: \tilde{X}^{j+1,j} \in \operatorname{Inj}(\mathcal{F})$ for $j \in \{0,\ldots,l-1\}$. It remains to establish isomorphisms of functors $\eta_{X^{i,j}} \colon h_{\tilde{X}^{i,j}} \cong \left(h^{X^{i,j}}\right)^*$ such that

- $e_{X^{i,j}}\left(- \circ \overline{\alpha^{i,j}}\right) = e_{X^{i,j+1}}\left(\overline{\tilde{\beta}^{i,j}} \circ -\right)$ on $\operatorname{Hom}_{\underline{\mathcal{F}}}(X^{i,j+1},\tilde{X}^{i,j})$ for all $0 \leq i \leq j < l$,
- $e_{X^{i,j}}\left(- \circ \overline{\beta^{i,j}}\right) = e_{X^{i+1,j}}\left(\overline{\tilde{\alpha}^{i,j}} \circ -\right)$ on $\operatorname{Hom}_{\underline{\mathcal{F}}}(X^{i+1,j},\tilde{X}^{i,j})$ for all $0 \leq i < j \leq l$.

We proceed inductively: Set $\eta_{X^{0,j}} := \hat{\eta}_{X^{0,j}}$ for $j \in \{0,\ldots,l\}$. Then $e_{X^{0,j}} = \hat{e}_{X^{0,j}}$ and the required properties hold. To construct $\eta_{X^{i+1,j+1}}$ for $i, j \in \{0,\ldots,l-1\}$ with $i \leq j$, suppose that $\eta_{X^{i',j'}}$ with the required properties is defined whenever $i' \leq i$ or $i' = i+1$ and $j' \leq j$. We may then assume that $\hat{X}^{i,j} = \tilde{X}^{i,j}$, $\hat{X}^{i,j+1} = \tilde{X}^{i,j+1}$, $\hat{X}^{i+1,j} = \tilde{X}^{i+1,j}$, $\overline{\hat{\alpha}^{i,j}} = \overline{\tilde{\alpha}^{i,j}}$, and $\overline{\hat{\beta}^{i,j}} = \overline{\tilde{\beta}^{i,j}}$. To complete the induction, apply Lemma 7.9 to the homotopy cartesian squares

$$\begin{tikzcd} X^{i,j} \arrow[r, "\overline{\alpha^{i,j}}"] \arrow[d, "\overline{\beta^{i,j}}"] \arrow[dr, phantom, "\square"] & X^{i,j+1} \arrow[d, "\overline{\beta^{i,j+1}}"] \\ X^{i+1,j} \arrow[r, "\overline{\alpha^{i+1,j}}"] & X^{i+1,j+1} \end{tikzcd} \qquad \begin{tikzcd} \tilde{X}^{i,j} \arrow[r, "\overline{\tilde{\beta}^{i,j}}"] \arrow[d, "\overline{\tilde{\alpha}^{i,j}}"] \arrow[dr, phantom, "\square"] & \tilde{X}^{i,j+1} \arrow[d, "\overline{\hat{\alpha}^{i,j+1}}"] \\ \tilde{X}^{i+1,j} \arrow[r, "\overline{\hat{\beta}^{i+1,j}}"] & \tilde{X}^{i+1,j+1} \end{tikzcd}$$

in $\underline{\mathcal{F}}$, see Proposition 2.9 and Lemma 2.17. □

Lemma 7.9. *In a triangulated category $\mathcal{T}$, hom-finite over a field, consider two homotopy cartesian squares:*

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "c"] \arrow[dr, phantom, "\square"] & B \arrow[d, "b"] & \tilde{A} \arrow[r, "\tilde{a}"] \arrow[d, "\tilde{c}"] \arrow[dr, phantom, "\square"] & \tilde{B} \arrow[d, "\tilde{b}"] \\
C \arrow[r, "d"] & D & \tilde{C} \arrow[r, "\tilde{d}"] & \tilde{D}
\end{tikzcd}$$

Suppose that there are isomorphisms of functors $\eta_A\colon h_{\tilde{A}} \cong \left(h^A\right)^*$, $\eta_B\colon h_{\tilde{B}} \cong \left(h^B\right)^*$, $\eta_C\colon h_{\tilde{C}} \cong \left(h^C\right)^*$, and $\hat{\eta}_D\colon h_{\hat{D}} \cong \left(h^D\right)^*$, where $\hat{D} \in \mathcal{T}$, such that

- $e_A(- \circ a) = e_B(\tilde{a} \circ -)$ *on* $\operatorname{Hom}(B, \tilde{A})$,
- $e_A(- \circ c) = e_C(\tilde{c} \circ -)$ *on* $\operatorname{Hom}(C, \tilde{A})$.

Then there is an isomorphism of functors $\eta_D\colon h_{\tilde{D}} \cong \left(h^D\right)^*$ such that

- $e_B(- \circ b) = e_D(\tilde{b} \circ -)$ *on* $\operatorname{Hom}(D, \tilde{B})$,
- $e_C(- \circ d) = e_D(\tilde{d} \circ -)$ *on* $\operatorname{Hom}(D, \tilde{C})$.

Proof. Set $\hat{e}_D := e_{\hat{\eta}_D} \in \left(h^D_{\hat{D}}\right)^*$ and consider the morphisms $\hat{b}\colon \tilde{B} \to \hat{D}$ and $\hat{d}\colon \tilde{C} \to \hat{D}$ such that $e_B(- \circ b) = \hat{e}_D(\hat{b} \circ -)$ on $\operatorname{Hom}(D, \tilde{B})$ and $e_C(- \circ d) = \hat{e}_D(\hat{d} \circ -)$ on $\operatorname{Hom}(D, \tilde{C})$, see Lemma 7.2. The second homotopy cartesian square and [BK89, Prop. 3.3] yield an isomorphism

$$\begin{tikzcd}
\tilde{A} \arrow[r, "\begin{pmatrix}\tilde{a}\\ -\tilde{c}\end{pmatrix}"] \arrow[d, equal] & \tilde{B} \oplus \tilde{C} \arrow[r, "\begin{pmatrix}\tilde{b} & \tilde{d}\end{pmatrix}"] \arrow[d, equal] & \tilde{D} \arrow[d, dashed, "f", "\cong"'] \\
\tilde{A} \arrow[r, "\begin{pmatrix}\tilde{a}\\ -\tilde{c}\end{pmatrix}"] & \tilde{B} \oplus \tilde{C} \arrow[r, "\begin{pmatrix}\hat{b} & \hat{d}\end{pmatrix}"] & \hat{D}
\end{tikzcd}$$

of distinguished triangles in $\mathcal{T}$. Define η_D as the composition

$$\begin{tikzcd}
h_{\tilde{D}} \arrow[rr, dashed, "\eta_D", "\cong"'] \arrow[dr, "\cong", "h_f"'] & & \left(h^D\right)^* & \operatorname{id}_{\tilde{D}} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \hat{e}_D(f \circ -) = e_D \\
& h_{\hat{D}}, \arrow[ur, "\cong", "\hat{\eta}_D"'] & & & f. \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

We have

- $e_B(- \circ b) = \hat{e}_D(\hat{b} \circ -) = \hat{e}_D(f\tilde{b} \circ -) = e_D(\tilde{b} \circ -)$ on $\operatorname{Hom}(D, \tilde{B})$,
- $e_C(- \circ d) = \hat{e}_D(\hat{d} \circ -) = \hat{e}_D(f\tilde{d} \circ -) = e_D(\tilde{d} \circ -)$ on $\operatorname{Hom}(D, \tilde{C})$,

as desired. □

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