THE STABILIZED BOUNDED N-DERIVED CATEGORY OF AN EXACT CATEGORY

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Abstract. Buchweitz related the singularity category of a (strongly) Gorenstein ring and the stable category of maximal Cohen-Macaulay modules by a triangle equivalence. We phrase his result in a relative categorical setting based on N-complexes instead of classical 2-complexes. The role of Cohen-Macaulay modules is played by chains of monics in a Frobenius subcategory of an exact category. As a byproduct, we provide foundational results on derived categories of N-complexes over exact categories known from the Abelian case or for 2-complexes.

Contents

2020 *Mathematics Subject Classification.* Primary 18G50, 18G80; Secondary 16E65, 18G35, 18G65.

Key words and phrases. N-complex, exact category, derived category, singularity category, Cohen-Macaulay.

JF would like to thank Janina C. Letz for helpful explanations.

INTRODUCTION

The Auslander-Buchsbaum-Serre theorem states that a Noetherian local ring S is regular if and only if all S-modules have a finite projective resolution. In 1980, Eisenbud [Eis80] showed that, over a complete intersection S, any minimal free resolution with bounded ranks becomes 2-periodic after a finite number of steps, depending only on $\dim(S)$.

In an unpublished preprint from 1986, Buchweitz captured this phenomenon in categorical language. To extract the asymptotic part of a complex he forms the Verdier quotient

$$\underline{\mathcal{D}}^b(S) = \mathcal{D}^b(S)/\mathrm{Perf}(S)$$

of the bounded derived category by complexes of projective modules, the *perfect* complexes. This *stabilized derived category* of a general (possibly non-commutative) ring S can be seen as a measure for its non-regularity. In 2004, it was rediscovered by Orlov [Orl04] in a geometric setting and is today known as the *singularity category*.

Buchweitz works over a (strongly) Gorenstein ring S. Beyond the (two-sided) injective dimension of S, non-zero syzygies of finite S-modules are maximal Cohen-Macaulay modules. Conversely, any such module can be considered a complex, when placed at different positions. This suggests an equivalence between $\underline{\mathcal{D}}^b(S)$ and the stable category $\underline{\mathrm{MCM}}(S)$ of maximal Cohen-Macaulay modules. Buchweitz confirmed this fact involving the category $\underline{\mathrm{APC}}(S)$ of complete resolutions. It serves both as a middleman between the two categories, who translates high to low syzygies and to compute Tate cohomology over S.

In 2021, Avramov, Briggs, Iyengar, and Letz [Buc21] published an annotated version of Buchweitz's manuscript. In Appendix B, they prove Buchweitz's result under weaker hypotheses, where any finite S-module is assumed to have a totally reflexive syzygy of order depending on the module.

Building on previous work of Murfet and Salarian [MS11] and others, Christensen et al. [Chr+23; CET20] generalized Buchweitz's theorem to schemes.

This article provides a purely categorical formulation of Buchweitz's theorem: We replace the category of finite S-modules by a general exact idempotent complete category $\mathcal{E}$ and the subcategory $\mathrm{MCM}(S)$ by a Frobenius subcategory $\mathcal{F}$, subject to a list of conditions. As an additional direction of generalization, we pass to a bounded derived category $\mathcal{D}^b_N(\mathcal{E})$ of *N-complexes*, where the Nth power of the differential is zero for $N \geq 2$. As a consequence, $\mathcal{F}$ becomes the category $\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{F})$ of chains of $N - 2$ many monics. Complete N-resolutions are then objects of the stable category $\underline{\mathrm{APC}}_N(\mathcal{F}) = \underline{\mathrm{TAPC}}_N(\mathcal{F})$ of (totally) acyclic N-complexes over $\mathcal{F}$ with projective objects. More specifically, our main result is

Theorem A. *Let $\mathcal{E}$ be an exact idempotent complete category and $\mathcal{F}$ a Frobenius category, which is a fully exact, replete subcategory of $\mathcal{E}$ with* $\mathrm{Proj}(\mathcal{F}) = \mathrm{Proj}(\mathcal{E})$. *Suppose that every object in $\mathcal{E}$ has a syzygy in $\mathcal{F}$. Then there is a commutative diagram of triangle equivalences*

$$\begin{tikzcd}
\underline{\mathrm{APC}}_N(\mathcal{F}) \arrow[rr, "\Omega^1", "\simeq"'] \arrow[dr, "\underline{\tau}^{\leq 0}"', "\simeq"] & & \underline{\mathrm{Mor}}^{\mathrm{m}}_{N-2}(\mathcal{F}) \arrow[dl, "\underline{\iota}^0", "\simeq"'] \\
& \underline{\mathcal{D}}^b_N(\mathcal{E}), &
\end{tikzcd}$$

where ι^0 is the embedding, $\tau^{\leq 0}$ the hard truncation and Ω^1 the N-syzygy, at the respective positions.

Our proof that $\underline{\tau}^{\leq 0}$ is an equivalence is inspired by the work of Orlov [Orl09].

Theorem A is known in special cases: Bahiraei, Hafezi, and Nematbakhsh [BHN16] establish an equivalence between $\underline{\mathrm{TAPC}}_N(S)$ and $\underline{\mathcal{D}}^b_N(S)$ over a left coherent ring S using triangular matrix rings. For $N = 2$, Christensen et al. prove the equivalences in Theorem A for a complete, hereditary cotorsion pair $(\mathcal{U}, \mathcal{V})$ in an Abelian category $\mathcal{A}$, see [Chr+23, Thm. 3.10] and [CET20, Thm. 3.8]. In their setup, $\mathcal{E} = \mathcal{V}$ and $\mathcal{F}$ is the subcategory of right $\mathcal{U}$-Gorenstein objects in $\mathcal{A}$ with $\mathrm{Proj}(\mathcal{F}) = \mathcal{U} \cap \mathcal{V} = \mathrm{Proj}(\mathcal{E})$, see [CET20, Thm. 2.11] and [Chr+23, Prop. 2.7.(a), Thm. 3.6]. Brightbill and Miemietz [BM24] prove Theorem A, without explicit mention of $\underline{\tau}^{\leq 0}$, under the assumption that $\mathcal{E}$ is a Gorenstein Abelian category and $\mathcal{F}$ its subcategory of Gorenstein projective objects.[1]

Already in 1942, N-complexes appeared in the work of Mayer [May42] generalizing simplicial homology theory. Kapranov [Kap96] and Dubois-Violette [Dub98] studied their homological properties. They find application in physics and other areas of mathematics, see for instance [DH99; CSW07; Hen08] and [Est07; GH10; KQ15]. Cassel and Wambst [KW98] studied N-resolutions, but only of single objects.

Theorem A requires foundations of N-derived categories over exact categories. Some have been laid by Iyama, Kato and Miyachi [IKM17] in the Abelian case, see also [YD15] and [YW15]. General results on semiorthogonal decompositions by Jørgensen and Kato [JK15] turn out particularly useful in this context. We combine this work with Keller's [Kel90; Kel96] on 2-derived categories of exact categories and rely on a generalization for deflation-exact categories due to Henrard and van Roosmalen [HR20]. In particular, we construct N-resolutions of bounded above N-complexes, see Subsection 2.4. This leads us to extend results of Verdier [Ver96, Ch. III, Thm. 1.2.3] and [IKM17, Thm. 3.12] in the following two theorems, formulated using Verdier's notation:

Theorem B. *For an exact idempotent complete category $\mathcal{E}$ there is a diagram of canonical fully faithful, triangle functors and equivalences:*

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[1] The final version of their work was published shortly before the completion of this article.

$$\begin{tikzcd}
 & & \mathcal{D}_N^{+}(\mathcal{E}) \arrow[dr, "\simeq"] & & \\
 & \mathcal{D}_N^{+,b}(\mathcal{E}) \arrow[ur] \arrow[dr, "\simeq"] & & \mathcal{D}_N^{\infty,+}(\mathcal{E}) \arrow[dr] & \\
\mathcal{D}_N^{b}(\mathcal{E}) \arrow[ur, "\simeq"] \arrow[dr, "\simeq"'] & & \mathcal{D}_N^{\infty,b}(\mathcal{E}) \arrow[ur] \arrow[dr] & & \mathcal{D}_N(\mathcal{E}) \\
 & \mathcal{D}_N^{-,b}(\mathcal{E}) \arrow[ur, "\simeq"'] \arrow[dr] & & \mathcal{D}_N^{\infty,-}(\mathcal{E}) \arrow[ur] & \\
 & & \mathcal{D}_N^{-}(\mathcal{E}) \arrow[ur, "\simeq"'] & &
\end{tikzcd}$$

Theorem C. *Let $\mathcal{E}$ be an exact idempotent complete category with enough projectives.*

(a) The pair $(\mathcal{K}_N^{-}(\mathrm{Proj}(\mathcal{E})), \mathcal{K}_N^{-,\varnothing}(\mathcal{E}))$ is a semiorthogonal decomposition of $\mathcal{K}_N^{-}(\mathcal{E})$, which gives rise to a triangle equivalence $\mathcal{D}_N^{-}(\mathcal{E}) \simeq \mathcal{K}_N^{-}(\mathrm{Proj}(\mathcal{E}))$.

(b) The pair $(\mathcal{K}_N^{-,b_{\mathcal{E}}}(\mathrm{Proj}(\mathcal{E})), \mathcal{K}_N^{-,\varnothing}(\mathcal{E}))$ is a semiorthogonal decomposition of $\mathcal{K}_N^{-,b}(\mathcal{E})$, which gives rise to a triangle equivalence $\mathcal{D}_N^{b}(\mathcal{E}) \simeq \mathcal{K}_N^{-,b_{\mathcal{E}}}(\mathrm{Proj}(\mathcal{E}))$.

The obvious dual statements hold as well.

Theorems A to C agree with Theorems 3.37, 3.9 and 3.11 of the main part.

1. Preliminaries

Unless stated otherwise, all (sub)categories and functors considered are assumed to be (full) additive.

Our main reference on the topic of *triangulated categories* is Neeman's book [Nee01]. However, we require the more general definition of a triangulated category whose suspension functor is only a autoequivalence instead of an automorphism. These two definitions agree up to a triangulated equivalence, see [KV87, §2] and [May01, §2].
Recall that a *triangle equivalence* is a triangulated functor which is an equivalence of categories. Its quasi-inverse is automatically a triangulated functor, see [BK89, Prop. 1.4] for a more general statement.

Unless stated otherwise, the image of a functor always means the full essential image. Given a fully faithful (triangulated) functor $F\colon C' \to C$, we tacitly identify C' up to equivalence with its image $F(C')$, which is a strictly full (triangulated) subcategory of C.

Bullets in diagrams represent arbitrary objects, and E_l denotes the unit matrix of size $l \in \mathbb{N}$.

1.1. Exact categories.

For convenience of the reader, we recollect relevant definitions and results on the topic of *exact categories* in the sense of Quillen [Qui73]. Our main reference is Bühler's expository article [Büh10], which relies in part on work of Keller [Kel90; Kel96].

Definition 1.1.

(a) A **pullback** (dashed) is the limit of a diagram (solid) of the form

$$\begin{tikzcd}
\bullet \arrow[r, dashed, "a'"] \arrow[d, dashed, "b'"] & \bullet \arrow[d, "b"] \\
\bullet \arrow[r, "a"] & \bullet.
\end{tikzcd}$$

(b) A **pushout** (dashed) is the colimit of a diagram (solid) of the form

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, "b"] & \bullet \arrow[d, dashed, "b'"] \\
\bullet \arrow[r, dashed, "a'"] & \bullet.
\end{tikzcd}$$

By abuse of wording, the morphism a' is called a **pullback**, resp. **pushout**, of a **along** b. The completed diagrams are also called a **pullback (square)**, resp. **pushout (square)**. A square is called **bicartesian** if it is both a pullback and a pushout.

Remark 1.2. If (X) and (Y) are two pullback, resp. pushout squares, then their concatenation (XY) is again a pullback, resp. pushout square.

For later reference, we mention the following converse of Remark 1.2:

Lemma 1.3. *Consider the following commutative diagram in an additive category:*

$$\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "a"] \arrow[dr, phantom, "(X)"] & B \arrow[r, "s"] \arrow[d, "b"] \arrow[dr, phantom, "(Y)"] & C \arrow[d, "c"] \\
A' \arrow[r, "r'"] & B' \arrow[r, "s'"] & C'.
\end{tikzcd}$$

(a) If the outer square (XY) is a pushout and $\begin{pmatrix} b & r' \end{pmatrix} : B \oplus A' \to B'$ is an epic, then (Y) is a pushout.

(b) If the outer square (XY) is a pullback and $\begin{pmatrix} s \\ b \end{pmatrix} : B \to C \oplus B'$ is a monic, then (X) is a pullback.

Proof.

(a) Consider the solid commutative diagram

$$\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "a"] & B \arrow[r, "s"] \arrow[d, "b"] & C \arrow[d, "c"] \arrow[ddr, "t"] & \\
A' \arrow[r, "r'"] \arrow[drrr, "t'r'"'] & B' \arrow[r, "s'"] \arrow[drr, "t'"'] & C' \arrow[dr, dashed, "u"] & \\
& & & P.
\end{tikzcd}$$

The outer pushout square yields a unique dashed morphism u with $uc = t$ and $us'r' = t'r'$, hence with $uc = t$ and $us'r' = t'r'$. Since $\begin{pmatrix} b & r' \end{pmatrix}$ is an epic, $us' = t'$ follows from

$$us' \begin{pmatrix} b & r' \end{pmatrix} = \begin{pmatrix} us'b & us'r' \end{pmatrix} = \begin{pmatrix} ucs & t'r' \end{pmatrix} = \begin{pmatrix} ts & t'r' \end{pmatrix} = \begin{pmatrix} t'b & t'r' \end{pmatrix} = t' \begin{pmatrix} b & r' \end{pmatrix}.$$

(b) is dual to (a). □

Definition 1.4 ([Büh10, Def. 2.1])**.** Let $\mathcal{S}$ be a collection of pairs (i, p) of morphisms in an additive category $\mathcal{E}$, where i is a kernel of p and p is a cokernel of i. If $(i, p) \in \mathcal{S}$, then i is called an **($\mathcal{S}$-)admissible monic** and p an **($\mathcal{S}$-)admissible epic**. The pairs $(i, p) \in \mathcal{S}$ are referred to as a **short ($\mathcal{S}$-)exact sequences** in $\mathcal{E}$ and are displayed as

$$A' \overset{i}{\rightarrowtail} A \overset{p}{\twoheadrightarrow} A''.$$

The collection $\mathcal{S}$ is said to define an **exact structure** on $\mathcal{E}$ if $\mathcal{S}$ is closed under isomorphisms and if the following axioms are satisfied:

(a) For all objects $A \in \mathcal{E}$, the identity id_A is an admissible monic and epic.
(b) Composition preserves admissible monics and epics.
(c) Pushout, resp. pullback, along arbitrary morphisms exists for and preserves admissible monics, resp. epics.

In this case, $(\mathcal{E}, \mathcal{S})$, or just $\mathcal{E}$, is called an **exact category**.

Remark 1.5 ([Büh10, Rem. 2.2]). If $\mathcal{E}$ is an exact category, then so is $\mathcal{E}^{\mathrm{op}}$ with admissible monics and epics exchanged. Therefore, each statement on exact structures has a dual. For the sake of clarity, we formulate some less obvious dual statements explicitly.

Definition 1.6. Let $\mathcal{A}$ be an additive category.

(a) A morphism in $\mathcal{A}$ is called a **split epic (monic)** if it has a right (left) inverse.
(b) A sequence of composable morphisms in $\mathcal{A}$ is called a **split short exact** if it is isomorphic to

$$A \longrightarrow A \oplus B \longrightarrow B$$

for $A, B \in \mathcal{A}$.

Remark 1.7 ([Büh10, Rem. 7.4]). If a split epic $r\colon Y \twoheadrightarrow Z$ in an additive category has a kernel $i\colon X \to Y$, then the sequence $X \xrightarrow{i} Y \xrightarrow{r} Z$ is split short exact.

Proposition 1.8 ([Büh10, Prop. 2.9])**.** *In an exact category, finite direct sums of short exact sequences are again short exact. In particular, any split short exact sequence is short exact.* □

Example 1.9.

(a) Any additive category has an exact structure given by the split short exact sequences. We refer to it as the **split** exact structure.
(b) Any Abelian category has the **maximal** exact structure with all monics and epics admissible.
(c) Any diagram category over an exact category $\mathcal{E}$ has an exact structure, defined component-wise by the exact structure of $\mathcal{E}$. We refer to it as the **termwise** exact structure.

Proposition 1.10 (Obscure axiom, [Büh10, Prop. 2.16])**.** *In an exact category, the following statements hold:*

(a) If a morphism $i\colon A \to B$ has a cokernel, and $b\colon B \to C$ is a morphism such that $bi\colon A \rightarrowtail C$ is an admissible monic, then i is an admissible monic.

(b) If a morphism $p\colon B \to C$ has a kernel, and $a\colon A \to B$ is a morphism such that $pa\colon A \twoheadrightarrow C$ is an admissible epic, then p is an admissible epic. □

Corollary 1.11 ([Büh10, Cor. 2.18]). *Let (i,p) and (i',p') be two pairs of composable morphisms in an exact category. If their direct sum $(i \oplus i', p \oplus p')$ is short exact, then both (i,p) and (i',p') are short exact.* □

Proposition 1.12 ([Büh10, Prop. 2.12]).

(a) For a square

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[d, "f'"] \\
A' \arrow[r, tail, "i'"] & B'
\end{tikzcd}$$

in an exact category, the following statements are equivalent:

(1) The square is a pushout.

(2) The square is bicartesian.

(3) The sequence $A \xrightarrow{\begin{pmatrix} i \\ -f \end{pmatrix}} B \oplus A' \xrightarrow{\begin{pmatrix} f' & i' \end{pmatrix}} B'$ *is short exact.*

(4) The square is part of a commutative diagram

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads] \arrow[d, "f'"] & C \arrow[d, equal] \\
A' \arrow[r, tail, "i'"] & B' \arrow[r, two heads] & C.
\end{tikzcd}$$

(b) For a square

$$\begin{tikzcd}
A \arrow[r, two heads, "p'"] \arrow[d, "g'"] & B \arrow[d, "g"] \\
A' \arrow[r, two heads, "p"] & B'
\end{tikzcd}$$

in an exact category, the following statements are equivalent:

(1) The square is a pullback.

(2) The square is bicartesian.

(3) The sequence $A \xrightarrow{\begin{pmatrix} p' \\ g' \end{pmatrix}} B \oplus A' \xrightarrow{\begin{pmatrix} -g & p \end{pmatrix}} B'$ *is short exact.*

(4) The square is part of a commutative diagram

$$\begin{tikzcd}
K \arrow[r, tail] \arrow[d, equal] & A \arrow[r, two heads, "p'"] \arrow[d, "g'"] & B \arrow[d, "g"] \\
K \arrow[r, tail] & A' \arrow[r, two heads, "p"] & B'.
\end{tikzcd}$$

□

Corollary 1.13. *In an exact category the following statements hold:*

(a) Pushouts of admissible monics and pullbacks of admissible epics are bicartesian squares.
(b) A square

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d, two heads] \\ C \arrow[r] & D \end{tikzcd}$$

is a pushout if and only if it is a pullback. In this case, opposite arrows are admissible morphisms of the same type.

Proof.

(a) For the pushout, combine Proposition 1.12.(a).(1) $\Leftrightarrow$ (2) with the pullback axiom, see Definition 1.4.(c). The argument for the pullback is dual.
(b) If the given diagram is a pushout, then it is a pullback by (a) and $A \twoheadrightarrow C$ is an admissible epic by the pushout axiom, see Definition 1.4.(c). The converse implication is dual. □

Proposition 1.14 ([Büh10, Prop. 2.15])**.** *In an exact category, pullback along an admissible epic preserves admissible monics and pushout along an admissible monic preserves admissible epics.* □

Lemma 1.15 (Noether lemma, [Büh10, Ex. 3.7])**.** *Consider the commutative diagram*

$$\begin{tikzcd} A' \arrow[r, tail] \arrow[d, tail] & B' \arrow[r, two heads] \arrow[d, tail] & C' \arrow[d, tail, dashed] \\ A \arrow[r, tail] \arrow[d, two heads] & B \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, two heads, dashed] \\ A'' \arrow[r, tail] & B'' \arrow[r, two heads] & C'' \end{tikzcd}$$

in an exact category, where the rows and solid columns are short exact sequences. Then there exist unique morphisms $C' \to C$ and $C \to C''$ making the diagram commute, and $C' \rightarrowtail C \twoheadrightarrow C''$ a short exact sequence. □

Definition 1.16. A covariant functor $F\colon (\mathcal{E}', \mathcal{S}') \to (\mathcal{E}, \mathcal{S})$ between exact categories is called **exact** if $(F(i), F(p)) \in \mathcal{S}$ for all $(i, p) \in \mathcal{S}'$. It is **fully exact** if, in addition, $(F(i), F(p)) \in \mathcal{S}$ implies $(i, p) \in \mathcal{S}'$, for all pairs (i, p) of composable morphisms in $\mathcal{E}'$. Obvious dual notions are defined for contravariant functors.

A subcategory $\mathcal{E}'$ of $\mathcal{E}$ is called **(fully) exact** if it is an exact category itself and the inclusion functor $\mathcal{E}' \to \mathcal{E}$ is (fully) exact.[2] Note that subcategories of additive categories are fully exact with respect to the split exact structure, see Example 1.9.(a).

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[2] Bühler uses the term *fully exact* for what we call *extension-closed*, see Definition 1.18.

Proposition 1.17 ([Büh10, Prop. 5.2])**.** *An exact functor preserves pushouts along admissible monics and pullbacks along admissible epics.* □

Definition 1.18 ([Büh10, Def. 10.21])**.** Let $\mathcal{E}'$ be a subcategory of an exact category $\mathcal{E}$. We call $\mathcal{E}'$ **extension-closed** if any $X \in \mathcal{E}$ which fits into a short exact sequence $Y' \rightarrowtail X \twoheadrightarrow Y$ with $Y, Y' \in \mathcal{E}'$ is an object of $\mathcal{E}'$.

Part (a) of Lemma 1.19 is [Büh10, Lem. 10.20], part (b) follows from Proposition 1.12:

Lemma 1.19. *A subcategory $\mathcal{E}'$ of an exact category $\mathcal{E}$ is fully exact if one of the following conditions holds:*

(a) $\mathcal{E}'$ is an extension-closed subcategory of $\mathcal{E}$.
(b) $\mathcal{E}'$ is closed in $\mathcal{E}$ under kernels of admissible epics and cokernels of admissible monics.

In both cases, the exact structure is given by short exact sequences in $\mathcal{E}$ with objects in $\mathcal{E}'$. □

Notation 1.20. The category of sequences over an additive category $\mathcal{A}$ is the diagram category $C(\mathcal{A}) := \operatorname{Func}(T, \mathcal{A})$ where T is the infinite linear quiver

$$\cdots \longrightarrow \underset{-1}{\bullet} \longrightarrow \underset{0}{\bullet} \longrightarrow \underset{1}{\bullet} \longrightarrow \cdots$$

with vertices indexed by $\mathbb{Z}$ in ascending order. We denote denote objects of $C(\mathcal{A})$ by $X = (X, d_X)$, where $X = (X^k)_{k\in\mathbb{Z}}$ and $d_X = (d_X^k)_{k\in\mathbb{Z}}$ with $d_X^k \colon X^k \to X^{k+1}$ for $k \in \mathbb{Z}$. We omit the index k of d_X^k when it is clear from the context. Incomplete sequences are extended by zeros without explicit mention.

Remark 1.21. Termwise finite coproducts of sequences over an additive category exist.

Remark 1.22. If $\mathcal{E}$ is an exact category, then $C(\mathcal{E})$ has two natural exact structures: The **termwise** exact structure from Example 1.9.(c) and the **termsplit** exact structure defined in the same way by the underlying additive category of $\mathcal{E}$, see Example 1.9.(a). Unless mentioned otherwise the termsplit exact structure is the *default* choice.
If $\mathcal{E}'$ is a (fully) exact subcategory of $\mathcal{E}$, then so is $C(\mathcal{E}')$ in $C(\mathcal{E}')$ due to the termwise exact structure. In particular, $C_N(\mathcal{A}')$ is fully exact in $C_N(\mathcal{A})$ for a subcategory $\mathcal{A}'$ of an additive category $\mathcal{A}$.

1.2. **Stable categories.**

Definition 1.23. An object P of an exact category $\mathcal{E}$ is called a **projective** if the covariant functor $\operatorname{Hom}_{\mathcal{E}}(P, -) \colon \mathcal{E} \to \mathrm{Ab}$ is exact. Dually, an object I of $\mathcal{E}$ is called an **injective** if the contravariant functor $\operatorname{Hom}_{\mathcal{E}}(-, I) \colon \mathcal{E} \to \mathrm{Ab}$ is exact. The respective subcategories of $\mathcal{E}$ are denoted by $\operatorname{Proj}(\mathcal{E})$ and $\operatorname{Inj}(\mathcal{E})$. An object is called a **projective-injective** if it is both projective and injective.

Proposition 1.24 ([Büh10, Prop. 11.3, Cor. 11.4])**.** *An object P of an exact category $\mathcal{E}$ is projective if and only if any one of the following equivalent conditions holds:*

(1) For any admissible epic $X \twoheadrightarrow Y$, any morphism $P \to Y$ lifts as follows:

$$\begin{tikzcd} & X \arrow[d, two heads] \\ P \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

(2) *The functor* $\operatorname{Hom}_{\mathcal{E}}(P, -)$ *sends admissible epics to surjections.*

(3) *Every admissible epic* $X \twoheadrightarrow P$ *splits.*

If a morphism $P \to Z$ *with* $P \in \operatorname{Proj}(\mathcal{E})$ *admits a right inverse, then* Z *is projective as well.* □

Remark 1.25. Due to Proposition 1.24.(3), Remark 1.7, and their duals, the subcategories $\operatorname{Proj}(\mathcal{E})$ and $\operatorname{Inj}(\mathcal{E})$ of an exact category $\mathcal{E}$ are closed under direct summands and extensions and thus fully exact, see Lemma 1.19.(a). Their exact structure is the split exact structure, see Example 1.9.(a).

Notation 1.26. The **projectively stable category** of an exact category $\mathcal{E}$ is denoted by $\underline{\mathcal{E}}$. It has the same objects as $\mathcal{E}$, and two morphisms agree if their difference factors through a projective. Dually, we denote the **injectively stable category** by $\overline{\mathcal{E}}$. These constructions have a universal property, see [Mac98, Prop. II.8.1].

Remark 1.27. In an exact category $\mathcal{E}$, the following statements hold:

(a) By Proposition 1.24.(1), a morphism $f\colon X \to Y$ in $\mathcal{E}$ is zero in $\underline{\mathcal{E}}$ if and only if it factors through any admissible epic $p_Y\colon P \twoheadrightarrow Y$ with $P \in \operatorname{Proj}(\mathcal{E})$. Dually, f is zero in $\overline{\mathcal{E}}$ if and only if it factors through any admissible monic $i_X\colon X \rightarrowtail I$ with $I \in \operatorname{Inj}(\mathcal{E})$.

(b) If $A \in \mathcal{E}$ with $A = 0$ in $\underline{\mathcal{E}}$, then $A \in \operatorname{Proj}(\mathcal{E})$. Indeed, $\mathrm{id}_A = 0$ in $\underline{\mathcal{E}}$ yields a $P \in \operatorname{Proj}(\mathcal{E})$ and a morphism $P \to A$ in $\mathcal{E}$ with right inverse. Then $A \in \operatorname{Proj}(\mathcal{E})$ due to the particular statement of Proposition 1.24.

Definition 1.28. We say that an exact subcategory $\mathcal{E}'$ of $\mathcal{E}$ has **enough $\mathcal{E}$-projectives** if there is an admissible epic $P \twoheadrightarrow X$ in $\mathcal{E}'$ with $P \in \operatorname{Proj}(\mathcal{E})$ for each $X \in \mathcal{E}'$. Having **enough $\mathcal{E}$-injectives** is defined dually. If this holds for $\mathcal{E}' = \mathcal{E}$, one says that $\mathcal{E}$ has enough **enough projectives**, resp. **enough injectives**.

Remark 1.29. An exact subcategory $\mathcal{E}'$ of $\mathcal{E}$ has enough $\mathcal{E}$-projectives if $\mathcal{E}'$ has enough projectives and $\operatorname{Proj}(\mathcal{E}') \subseteq \operatorname{Proj}(\mathcal{E})$.

Definition 1.30. A **Frobenius (exact) category** is an exact category $\mathcal{F}$ with enough injectives, enough projectives, and $\operatorname{Proj}(\mathcal{F}) = \operatorname{Inj}(\mathcal{F})$. In this case, $\underline{\mathcal{F}} = \overline{\mathcal{F}}$ is called the **stable category** of $\mathcal{F}$. By a **sub-Frobenius category** of a Frobenius category $\mathcal{F}$, we mean an exact subcategory $\mathcal{F}'$ which has enough $\mathcal{F}$-projectives and enough $\mathcal{F}$-injectives. This terminology is justified by Lemma 1.36.(b).

Construction 1.31. Let $\mathcal{E}$ be an exact category with enough injectives. For each $X \in \mathcal{E}$, pick an admissible monic $i = i_X\colon X \rightarrowtail I(X)$ with $I(X) \in \operatorname{Inj}(\mathcal{E})$ and cokernel denoted by ΣX. If $f\colon X \to Y$ is a morphism in $\mathcal{E}$, then Proposition 1.12.(a) yields a pushout diagram

$$\begin{tikzcd} X \arrow[r, tail, "i"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & I(X) \arrow[r, two heads] \arrow[d, "f'"] & \Sigma X \arrow[d, equal] \\ Y \arrow[r, tail, "i'"] & C(f) \arrow[r, two heads] & \Sigma X \end{tikzcd} \tag{$D(f)$}$$

and a short exact sequence

$$\begin{tikzcd}[column sep=large] X \arrow[r, tail, "\begin{pmatrix} i \\ -f \end{pmatrix}"] & I(X) \oplus Y \arrow[r, two heads, "\begin{pmatrix} f' & i' \end{pmatrix}"] & C(f). \end{tikzcd} \tag{$S(f)$}$$

The object $C(f) \in \mathcal{E}$ is called the **(mapping) cone** of f. A **standard triangle** in $\underline{\mathcal{E}}$ is any sequence of the form

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r] & C(f) \arrow[r] & \Sigma X. \end{tikzcd} \tag{$T(f)$}$$

Dually, if $\mathcal{E}$ has enough projectives, the **(mapping) cocone** $C^*(f)$ of f fits into a pullback diagram

$$\begin{tikzcd} \Sigma^{-1} Y \arrow[r, tail] \arrow[d, equal] & C^*(f) \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ \Sigma^{-1} Y \arrow[r, tail] & P(Y) \arrow[r, two heads, "p"] & Y \end{tikzcd} \tag{$D^*(f)$}$$

and into a short exact sequence

$$\begin{tikzcd} C^*(f) \arrow[r, tail] & X \oplus P(Y) \arrow[r, two heads, "\begin{pmatrix} -f & p \end{pmatrix}"] & Y, \end{tikzcd} \tag{$S^*(f)$}$$

where $p = p_Y : P(Y) \twoheadrightarrow Y$ is an admissible epic with $P(Y) \in \operatorname{Proj}(\mathcal{E})$, and $\Sigma^{-1}Y$ denotes the cokernel of p. Note that $\Sigma X \cong C(X \to 0)$ and $\Sigma^{-1} Y \cong C^*(0 \to Y)$ in $\underline{\mathcal{E}}$.

Theorem 1.32 ([Hap88, Thm. 2.6]). *The stable category of a Frobenius category is triangulated. Its suspension functor Σ and the quasi-inverse Σ^{-1} are defined as in Construction 1.31. The distinguished triangles are those candidate triangles isomorphic to standard triangles.* □

Lemma 1.33 ([Hap88, Lem. 2.7]). *Any short exact sequence $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ in a Frobenius category $\mathcal{F}$ induces a distinguished triangle $X \xrightarrow{i} Y \xrightarrow{p} Z \longrightarrow \Sigma X$ in $\underline{\mathcal{F}}$.* □

Remark 1.34. Rotating the distinguished triangle obtained from ($S^*(f)$) by Lemma 1.33 and comparing with ($T(f)$) yields an isomorphism $\Sigma C^*(f) \cong C(f)$ in $\underline{\mathcal{F}}$.

Proposition 1.35 ([IKM16, Prop. 7.3]). *If $F : \mathcal{F}' \to \mathcal{F}$ is an exact functor between Frobenius categories which preserves projective-injectives*[3]*, then the induced functor $\underline{F} : \underline{\mathcal{F}'} \to \underline{\mathcal{F}}$ is triangulated.* □

Lemma 1.36. *Let $\mathcal{E}'$ be an exact subcategory of $\mathcal{E}$.*

---

[3] The hypothesis can be restricted to the objects of the form $I(X)$ and $P(Y)$ from Construction 1.31.

(a) *If $\mathcal{E}'$ has enough $\mathcal{E}$-projectives, then* $\operatorname{Proj}(\mathcal{E}') = \operatorname{Proj}(\mathcal{E}) \cap \mathcal{E}'$, *and the canonical functor* $\underline{\mathcal{E}'} \to \underline{\mathcal{E}}$ *is fully faithful.*

(b) *If $\mathcal{F}'$ is a sub-Frobenius category of $\mathcal{F}$, then $\mathcal{F}'$ is Frobenius, and the canonical functor $\underline{\mathcal{F}'} \to \underline{\mathcal{F}}$ is fully faithful and triangulated.*

Proof.

(a) Let $X \in \operatorname{Proj}(\mathcal{E}')$ and pick an admissible epic $p \colon P \twoheadrightarrow X$ in $\mathcal{E}'$ with $P \in \operatorname{Proj}(\mathcal{E})$. Then p has a right inverse due to Proposition 1.24.(1) applied to $X \in \operatorname{Proj}(\mathcal{E}')$ and id_X. We obtain $X \in \operatorname{Proj}(\mathcal{E})$ by the particular statement of Proposition 1.24. The converse inclusion holds trivially by Proposition 1.24.(1). Now $\mathcal{E}' \subseteq \mathcal{E}$ induces a full functor $\underline{\mathcal{E}'} \to \underline{\mathcal{E}}$. For faithfulness, consider a morphism $f \colon X \to Y$ in $\mathcal{E}'$ which is zero in $\underline{\mathcal{E}}$. By assumption, there is an admissible epic $p \colon P \twoheadrightarrow Y$ in $\mathcal{E}$ with $P \in \operatorname{Proj}(\mathcal{E}) \cap \mathcal{E}' = \operatorname{Proj}(\mathcal{E}')$. Then f factors through p and is zero in $\underline{\mathcal{E}'}$, see Remark 1.27.(a).

(b) By (a), $\mathcal{F}'$ is Frobenius, and $\mathcal{F}' \subseteq \mathcal{F}$ induces a fully faithful functor $\underline{\mathcal{F}} \to \underline{\mathcal{F}'}$. It is triangulated by Proposition 1.35. □

1.3. **Acyclicity and syzygies.**

Definition 1.37 ([Büh10, Def. 8.1])**.** An **admissible** morphism in an exact category is the composition of an admissible epic and an admissible monic, displayed as

$$\begin{tikzcd}
X \arrow[rr, "\circ" marking] \arrow[dr, two heads] & & Y \\
& I. \arrow[ur, tail] &
\end{tikzcd}$$

Remark 1.38. The admissible morphisms, which are monics (epics), are exactly the admissible monics (epics) in Definition 1.4.

Remark 1.39 ([Büh10, Lem. 8.4, Rem. 8.5, Ex. 8.6]).

(a) The defining factorization of an admissible morphism is unique up to a unique isomorphism.

(b) Any admissible morphism f has a so-called **analysis**

$$\begin{tikzcd}
& X \arrow[rr, "f", "\circ" marking] \arrow[dr, two heads, "e"] & & Y \arrow[dr, two heads, "c"] & \\
Z \arrow[ur, tail, "k"] & & I \arrow[ur, tail, "m"] & & C,
\end{tikzcd}$$

where k is a kernel, c is a cokernel, e is a coimage, and m is an image of f. In particular, all these morphisms are uniquely determined by f up to a unique isomorphism.

(c) Only for Abelian categories the class of admissible morphisms is closed under composition.

Definition 1.40 ([Büh10, Def. 8.8])**.** Let $\mathcal{E}$ be an exact category.

(a) A sequence $X' \longrightarrow X \longrightarrow X''$ of two morphisms in $\mathcal{E}$ is called **acyclic** if both are admissible morphisms and their factorizations

$$\begin{tikzcd}
X' \arrow[rr, "\circ" marking] \arrow[dr, two heads] & & X \arrow[rr, "\circ" marking] \arrow[dr, two heads] & & X'' \\
& Z \arrow[ur, tail] & & C \arrow[ur, tail] &
\end{tikzcd}$$

give rise to a short exact sequence $Z \rightarrowtail X \twoheadrightarrow C$.

(b) A sequence $X = (X^k)_{k\in\mathbb{Z}} \in C(\mathcal{E})$ is called **2-acyclic at position** $n \in \mathbb{Z}$ if the sequence $X^{n-1} \longrightarrow X^n \longrightarrow X^{n+1}$ is acyclic in the sense of (a). It is called **2-acyclic** if it is 2-acyclic at all positions $n \in \mathbb{Z}$.

Definition 1.41. A **projective resolution** of an object $X \in \mathcal{E}$ is a sequence $P \in C(\mathcal{E})$ with $P^k \in \operatorname{Proj}(\mathcal{E})$ for $k \in \mathbb{Z}_{\leq 0}$ and $P = 0$ for $k \in \mathbb{Z}_{>0}$ which fits into a 2-acyclic sequence

$$\begin{tikzcd}
P\colon \cdots \arrow[r, "\circ" marking] & P^{-n} \arrow[rr, "\circ" marking] \arrow[dr, two heads] & & P^{-n-1} \arrow[r, "\circ" marking] & \cdots \arrow[r, "\circ" marking] & P^{-1} \arrow[rr, "\circ" marking] \arrow[dr, two heads] & & P^0 \arrow[rr, two heads] \arrow[dr, two heads] & & X \\
& & C^{-n} \arrow[ur, tail] & & & & C^{-1} \arrow[ur, tail] & & C^0. \arrow[ur, equal] &
\end{tikzcd}$$

It exists for $X \in \mathcal{E}$ if $\mathcal{E}$ has enough projectives, see [Büh10, Prop. 12.2]. We refer to the object $\operatorname{syz}^n_P(X) := C^{-n}$ as an ***n*th syzygy** of X, for any $n \in \mathbb{N}$.

As in the Abelian case, Schanuel's Lemma holds in any exact category, and iterated application yields the well-definedness of syzygies up to projective equivalence.

Lemma 1.42 (Schanuel's Lemma). *Consider two short exact sequences*

$$Z \overset{i}{\rightarrowtail} P \overset{p}{\twoheadrightarrow} X \qquad \text{and} \qquad Z' \overset{i'}{\rightarrowtail} P' \overset{p'}{\twoheadrightarrow} X$$

in an exact category $\mathcal{E}$ with $P, P' \in \operatorname{Proj}(\mathcal{E})$. Then $Z \oplus P' \cong Z' \oplus P$.

Proof. Due to Proposition 1.24.(1), we obtain a commutative diagram

$$\begin{tikzcd}
Z \arrow[r, tail, "i"] \arrow[d, dashed, "b"] & P \arrow[r, two heads, "p"] \arrow[d, dashed, "a"] & X \arrow[d, equal] \\
Z' \arrow[r, tail, "i'"] & P' \arrow[r, two heads, "p'"] & X.
\end{tikzcd}$$

Then Proposition 1.12.(a) yields a short exact sequence $Z \rightarrowtail Z' \oplus P \twoheadrightarrow P'$ and the claim follows from Proposition 1.24.(3) and Remark 1.7.[4] □

Proposition 1.43. *Let $\mathcal{E}$ be an exact category. If P and Q are two projective resolutions of $X \in \mathcal{E}$, then $\operatorname{syz}^n_P(X) \oplus \tilde{Q}^n \cong \operatorname{syz}^n_Q(X) \oplus \tilde{P}^n$ for any $n \in \mathbb{N}$ and suitable $\tilde{P}^n, \tilde{Q}^n \in \operatorname{Proj}(\mathcal{E})$.* □

1.4. **Categories of N-complexes.** In this subsection, we review some foundational results on N-complexes by Iyama, Kato and Miyachi [IKM17].

Definition 1.44. Let $N \in \mathbb{N}$ with $N \geq 2$. A sequence $X \in C(\mathcal{A})$ over an additive category $\mathcal{A}$ is called an ***N*-complex** if the N-fold composition $d_X^{n+N-1} \circ \cdots \circ d_X^n$ is zero for all $n \in \mathbb{Z}$. The subcategory of $C(\mathcal{A})$ consisting of N-complexes is denoted by $C_N(\mathcal{A})$.

---

[4] A proof of the dual statement can be found in [MR22, Prop. 3.1].

Remark 1.45. If $\mathcal{E}$ is an exact category and $X \rightarrowtail Y \twoheadrightarrow Z$ is a termwise short exact sequence in $C(\mathcal{E})$ with $Y \in C_N(\mathcal{E})$, then also $X, Z \in C_N(\mathcal{E})$ by pre- and postcomposition. So, $C_N(\mathcal{E})$ is a fully exact subcategory of $C(\mathcal{E})$ with the termwise exact structure by Lemma 1.19.(b). In particular, $C_N(\mathcal{E})$ is a fully exact subcategory of $C(\mathcal{E})$ with its default exact structure, see Remark 1.22.
On the fully exact subcategory $C_N(\mathrm{Proj}(\mathcal{E}))$ of $C_N(\mathcal{E})$, both the termwise and the termsplit exact structure coincide with the exact structure as a fully exact subcategory of $C(\mathrm{Proj}(\mathcal{E}))$, see Remark 1.25. This holds verbatim for $C_N(\mathrm{Inj}(\mathcal{E}))$.

Notation 1.46. Given an object $A \in \mathcal{A}$ of an additive category $\mathcal{A}$, the covariant Hom functor $\mathrm{Hom}_{\mathcal{A}}(A, -) : C(\mathcal{A}) \to C(\mathrm{Ab})$ is defined by $\mathrm{Hom}_{\mathcal{A}}(A, X)^n := \mathrm{Hom}_{\mathcal{A}}(A, X^n)$ for $X \in C(\mathcal{A})$ and $n \in \mathbb{Z}$. Dually, the contravariant Hom functor is defined by $\mathrm{Hom}_{\mathcal{A}}(X, A)^n := \mathrm{Hom}_{\mathcal{A}}(X^{-n}, A)$.

Notation 1.47 ([IKM17, §2])**.** Let $\mathcal{A}$ be an additive category. For $A \in \mathcal{A}$, $s \in \mathbb{Z}$, and $t \in \{1, \dots, N-1\}$, the N-complex

$$\mu_t^s(A)\colon \ \cdots \longrightarrow 0 \longrightarrow A^{s-t+1} \xrightarrow{\mathrm{id}_A} \cdots \xrightarrow{\mathrm{id}_A} A^{s-1} \xrightarrow{\mathrm{id}_A} A^s \longrightarrow 0 \longrightarrow \cdots$$

is defined by $A^k = A$ for all $k \in \{s-t+1, \dots, s\}$.

Remark 1.48. Let $\mathcal{A}$ be an additive category. For $A, B \in \mathcal{A}$, $s \in \mathbb{Z}$, and $t \in \{1, \dots, N-1\}$, we have $\mathrm{Hom}_{\mathcal{A}}(B, \mu_t^s(A)) = \mu_t^s(\mathrm{Hom}_{\mathcal{A}}(B, A))$ and $\mathrm{Hom}_{\mathcal{A}}(\mu_t^s(A), B) = \mu_t^{-s+t-1}(\mathrm{Hom}_{\mathcal{A}}(A, B))$.

Remark 1.49 ([IKM17, (2.1)]). For any N-complex $X \in C_N(\mathcal{A})$, $A \in \mathcal{A}$, and $s \in \mathbb{Z}$, there are functorial isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(A, X^s) \cong \mathrm{Hom}_{C_N(\mathcal{A})}(\mu_N^{s+N-1}(A), X) \quad \text{and} \quad \mathrm{Hom}_{\mathcal{A}}(X^s, A) \cong \mathrm{Hom}_{C_N(\mathcal{A})}(X, \mu_N^s(A))$$

of Abelian groups given by mapping $f \in \mathrm{Hom}_{\mathcal{A}}(A, X^s)$ and $g \in \mathrm{Hom}_{\mathcal{A}}(X^s, A)$ to p_f^s and i_g^s, respectively, as depicted in the following commutative diagram:

$$\begin{tikzcd}
\mu_N^{s+N-1}(A)\colon \arrow[d, "p_f^s"] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r, "\mathrm{id}_A"] \arrow[d, "f"] & \cdots \arrow[r, "\mathrm{id}_A"] & A \arrow[r] \arrow[d, "d_X^{\{N-1\}} f"] & 0 \arrow[r] \arrow[d] & \cdots \\
X\colon & \cdots \arrow[r, "d_X"] & X^{s-1} \arrow[r, "d_X"] & X^s \arrow[r, "d_X"] & \cdots \arrow[r, "d_X"] & X^{s+N-1} \arrow[r, "d_X"] & X^{s+N} \arrow[r, "d_X"] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
X\colon \arrow[d, "i_g^s"] & \cdots \arrow[r, "d_X"] & X^{s-N} \arrow[r, "d_X"] \arrow[d] & X^{s-N+1} \arrow[r, "d_X"] \arrow[d, "d_X^{\{N-1\}} g"] & \cdots \arrow[r, "d_X"] & X^s \arrow[r, "d_X"] \arrow[d, "g"] & X^{s+1} \arrow[r, "d_X"] \arrow[d] & \cdots \\
\mu_N^s(A)\colon & \cdots \arrow[r] & 0 \arrow[r] & A \arrow[r, "\mathrm{id}_A"] & \cdots \arrow[r, "\mathrm{id}_A"] & A \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Note that $\mathrm{Hom}_{\mathcal{A}}(i_g^s, B) = p_{\mathrm{Hom}_{\mathcal{A}}(g,B)}^{-s}$ and $\mathrm{Hom}_{\mathcal{A}}(p_f^s, B) = i_{\mathrm{Hom}_{\mathcal{A}}(f,B)}^{-s}$ for any $B \in \mathcal{A}$.

Lemma 1.50 ([IKM17, Lem. 2.2])**.** *For an additive category $\mathcal{A}$, the N-complexes of the form $\mu_t^s(A)$ are projective-injectives of $C_N(\mathcal{A})$ for each $A \in \mathcal{A}$, $s \in \mathbb{Z}$, and $t \in \{1, \dots, N-1\}$.*

Construction 1.51 ([IKM17, (2.2)])**.** For any additive category $\mathcal{A}$, the exact category $C_N(\mathcal{A})$ has enough projectives and injectives: For each $X \in C_N(\mathcal{A})$, there is an

- admissible monic $i_X := \left(i^k_{\mathrm{id}_{X^k}}\right)_{k\in\mathbb{Z}} : X \rightarrowtail I(X)$ with $I(X) = I_N(X) := \bigoplus_{k\in\mathbb{Z}} \mu^k_N(X^k)$ injective, and an
- admissible epic $p_X := \left(p^k_{\mathrm{id}_{X^k}}\right)_{k\in\mathbb{Z}} : P(X) \twoheadrightarrow X$ with $P(X) = P_N(X) := \bigoplus_{k\in\mathbb{Z}} \mu^k_N(X^{k-N+1})$ projective,

see Remarks 1.21 and 1.49 and Lemma 1.50. These are our default choices for Construction 1.31.

Remark 1.52. Note that I and P, as defined in Construction 1.51, are functorial: If $f : X \to Y$ is a morphism of N-complexes over an exact category $\mathcal{E}$, then the component $I(f)^n$, resp. $P(f)^n$, of the lift is given on P^k by f^k, where $k \in \{n, \dots, n+N-1\}$, resp. $k \in \{n-N+1, \dots, n\}$.

Remark 1.53. For an N-complex $X \in C_N(\mathcal{A})$ over an additive category $\mathcal{A}$ and $A \in \mathcal{A}$, we have in $C_N(\mathrm{Ab})$:

(a) $\mathrm{Hom}_{\mathcal{A}}(i_X, A) = p_{\mathrm{Hom}_{\mathcal{A}}(X,A)}$ and $\mathrm{Hom}_{\mathcal{A}}(p_X, A) = i_{\mathrm{Hom}_{\mathcal{A}}(X,A)}$, see Remark 1.49.

(b) In particular, applying $\mathrm{Hom}_{\mathcal{A}}(-, A)$ to the sequences $(S(f))$ and $(S^*(f))$ yields

$$\mathrm{Hom}_{\mathcal{A}}(C(f), A) \cong C^*(\mathrm{Hom}_{\mathcal{A}}(f, A)) \text{ and } \mathrm{Hom}_{\mathcal{A}}(C^*(f), A) \cong C(\mathrm{Hom}_{\mathcal{A}}(f, A)).$$

Theorem 1.54 ([Hap88, p. 28], [IKM17, Thm. 2.1])**.** *The exact category $C_N(\mathcal{A})$ of N-complexes over an additive category $\mathcal{A}$ is Frobenius (with the termsplit exact structure).* □

Notation 1.55. The stable category of $C_N(\mathcal{A})$ is denoted by $\mathcal{K}_N(\mathcal{A})$. This is a triangulated category, see Theorems 1.32 and 1.54.

Remark 1.56. Let $\mathcal{A}'$ be a subcategory of an additive category $\mathcal{A}$. Due to Remarks 1.22 and 1.29, Construction 1.51, and Theorem 1.54, all assumptions in Lemma 1.36.(b) are satisfied, and $C_N(\mathcal{A}') \subseteq C_N(\mathcal{A})$ induces a fully faithful, triangulated functor $\mathcal{K}_N(\mathcal{A}') \to \mathcal{K}_N(\mathcal{A})$.

Notation 1.57. Let $X \in C(\mathcal{A})$ be a sequence over an additive category $\mathcal{A}$.

- We write $d^{\{r\}}_X = (d^{n+r-1}_X \circ \cdots \circ d^n_X)_{n\in\mathbb{Z}}$ for the rth power of d_X.
- The **shift functor** $\Theta : C(\mathcal{A}) \to C(\mathcal{A})$ is given by $(\Theta X)^k := X^{k+1}$ and $d^k_{\Theta X} := d^{k+1}_X$.
- By $-X$ we denote the sequence with $(-X)^k := X^k$ and $d^k_{-X} := -d^k_X$.

Construction 1.58. Let $f\colon X \to Y$ be a morphism of N-complexes over an additive category $\mathcal{A}$.

(a) In [IKM17, 693ff.], there is an explicit description of the cone $C(f)$ and the cocone $C^*(f)$ and their special cases, the suspension functor Σ and its quasi-inverse Σ^{-1}:

$$(\Sigma X)^n = \bigoplus_{k=1}^{N-1} X^{n+k}, \qquad d_{\Sigma X} = \left(\begin{array}{c|ccccc} 0 & & & E_{N-2} & & \\ \hline -d^{\{N-1\}}_X & -d^{\{N-2\}}_X & -d^{\{N-3\}}_X & \cdots & -d^{\{2\}}_X & -d_X \end{array}\right),$$

$$(\Sigma^{-1} X)^n = \bigoplus_{k=-N+1}^{-1} X^{n+k}, \qquad d_{\Sigma^{-1} X} = \left(\begin{array}{c|c} -d_X & \\ -d^{\{2\}}_X & \\ \vdots & E_{N-2} \\ -d^{\{N-3\}}_X & \\ -d^{\{N-2\}}_X & \\ \hline -d^{\{N-1\}}_X & 0 \end{array}\right),$$

$$C(f)^n = Y^n \oplus \bigoplus_{k=1}^{N-1} X^{n+k}, \qquad d_{C(f)} = \left(\begin{array}{c|ccccc} d_Y & f & 0 & \cdots & 0 & 0 \\ \hline 0 & & & d_{\Sigma X} & & \end{array}\right),$$

$$C^*(f)^n = \bigoplus_{k=-N+1}^{-1} Y^{n+k} \oplus X^n, \qquad d_{C^*(f)} = \left(\begin{array}{c|c} & 0 \\ & 0 \\ d_{\Sigma^{-1}Y} & \vdots \\ & 0 \\ & -f \\ \hline 0 & d_X \end{array}\right).$$

(b) Note that $C(X \to 0) = \Sigma X$ and $C^*(0 \to Y) = \Sigma^{-1} Y$, see Construction 1.31.

(c) Comparing cone and cocone in the case $N = 2$ yields $\Theta C^*(f) = -C(f)$, see Remark 1.34.

Theorem 1.59 ([IKM17, Thm. 2.4]). *There is a functorial isomorphism* $\Sigma^2 \cong \Theta^N$ *of endofunctors on* $\mathcal{K}_N(\mathcal{A})$*, for any additive category* $\mathcal{A}$*.*

Lemma 1.60 ([IKM17, Lem. 2.6 (i)]). *Let* $\mathcal{A}$ *be an additive category. For all* $A \in \mathcal{A}$*,* $k, s \in \mathbb{Z}$*,* $t \in \{1, \ldots, N-1\}$*, and* $l \in \{0, 1\}$*, we have*

$$\Sigma^{2k+l} \mu_t^s(A) = \begin{cases} \mu_t^{-kN+s}(A), & \text{if } l = 0, \\ \mu_{N-t}^{-kN+s-t}(A), & \text{if } l = 1. \end{cases}$$

Notation 1.61. Given a sequence $X \in C(\mathcal{A})$ over an additive category $\mathcal{A}$, $r \in \mathbb{N}$, and $n \in \mathbb{Z}$, set

- $Z^n_{(r)} := Z^n_{(r)}(X) := \ker\left(d_X^{n+r-1} \circ \cdots \circ d_X^n\right)$,
- $B^n_{(r)} := B^n_{(r)}(X) := \operatorname{im}\left(d_X^{n-1} \circ \cdots \circ d_X^{n-r}\right)$,
- $C^n_{(r)} := C^n_{(r)}(X) := \operatorname{coker}\left(d_X^{n-1} \circ \cdots \circ d_X^{n-r}\right)$,

if the respective object exists. This is the case, for instance, if the respective r-fold composition is an admissible morphism, see Definition 1.37. If $\mathcal{A}$ is an Abelian category, the homology of an N-complex $X \in C_N(\mathcal{A})$ is defined as

$$H^n_{(r)} := H^n_{(r)}(X) := Z^n_{(r)}(X)/B^n_{(N-r)}(X).$$

The lower index $r = 1$ is omitted if $N = 2$.

Remark 1.62. For an object A of an additive category $\mathcal{A}$,

$$B^n_{(N-r)}(\mu_N^s(A)) = Z^n_{(r)}(\mu_N^s(A)) = \begin{cases} A, & \text{if } s - r + 1 \leq n \leq s, \\ 0, & \text{otherwise,} \end{cases}$$

$$C^n_{(r)}(\mu_N^s(A)) = \begin{cases} A, & \text{if } s - N + 1 \leq n \leq s - N + r, \\ 0, & \text{otherwise,} \end{cases}$$

for all $n, s \in \mathbb{Z}$ and $r \in \{1, \ldots, N-1\}$.

Remark 1.63 ([IKM17, (3.4)]). For an N-complex $X \in C_N(\mathcal{A})$ over an additive category $\mathcal{A}$, and any $A \in \mathcal{A}$, there are the following isomorphisms of Abelian groups for all $s \in \mathbb{Z}$ and $t \in \{1, \ldots, N-1\}$:

(a) $\operatorname{Hom}_{C_N(\mathcal{A})}(\mu_t^s(A), X) \cong Z_{(t)}^{s-t+1}(\operatorname{Hom}_{\mathcal{A}}(A, X))$
(b) $\operatorname{Hom}_{\mathcal{K}_N(\mathcal{A})}(\mu_t^s(A), X) \cong H_{(t)}^{s-t+1}(\operatorname{Hom}_{\mathcal{A}}(A, X))$
(c) $\operatorname{Hom}_{C_N(\mathcal{A})}(X, \mu_t^s(A)) \cong Z_{(t)}^{-s}(\operatorname{Hom}_{\mathcal{A}}(X, A))$
(d) $\operatorname{Hom}_{\mathcal{K}_N(\mathcal{A})}(X, \mu_t^s(A)) \cong H_{(t)}^{-s}(\operatorname{Hom}_{\mathcal{A}}(X, A))$

Definition 1.64. A morphism $f \colon X \to Y$ of N-complexes over an additive category $\mathcal{A}$ is called **(N-)null-homotopic** if there exists a collection $h = (h^k)_{k\in\mathbb{Z}}$ of morphisms $h^k \colon X^k \to Y^{k-N+1}$ in $\mathcal{A}$ such that

$$f^k = \sum_{r=0}^{N-1} d_Y^{\{N-r-1\}} h^{k+r} d_X^{\{r\}}$$

for all $k \in \mathbb{Z}$. Such an h is referred to as an **(N-)homotopy**. Two morphisms in $C_N(\mathcal{A})$ are called **(N-)homotopy equivalent** if their difference is N-null-homotopic. The **homotopy category** of $C_N(\mathcal{A})$ has the same objects as $C_N(\mathcal{A})$ and N-homotopy equivalence classes as morphisms.

Theorem 1.65 ([IKM17, Thm. 2.3])**.** *The homotopy category of $C_N(\mathcal{A})$ over an additive category $\mathcal{A}$ is the stable category $\mathcal{K}_N(\mathcal{A})$.*

Definition 1.66. An N-complex $X \in C_N(\mathcal{A})$ over an additive category $\mathcal{A}$ is called **(N-)null-homotopic** or **(N-)contractible** if id_X is N-null-homotopic, or equivalently, if $X = 0$ in $\mathcal{K}_N(\mathcal{E})$.

1.5. **Categories of monics.** Categories of morphisms have been studied by Brightbill and Miemietz [BM24, §3]. We collect some of their results, see also [Büh10, Ex. 13.12].

Notation 1.67 ([BM24, Def. 3.1], [IKM17, Def 4.1])**.** For an additive category $\mathcal{A}$ and $l \in \mathbb{N}$, let $\operatorname{Mor}_l(\mathcal{A})$ denote the diagram category $\operatorname{Func}(T_l, \mathcal{A})$ where T_l is the linear quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{l}{\bullet} \arrow[r] & \underset{l+1}{\bullet} \end{tikzcd}$$

with l arrows. Over an exact category $\mathcal{E}$, we denote by

- $\operatorname{Mor}_l^{\mathrm{m}}(\mathcal{E})$ the subcategory of $\operatorname{Mor}_l(\mathcal{E})$ where all arrows map to admissible monics,
- $\operatorname{Mor}_l^{\mathrm{sm}}(\mathcal{E})$ the subcategory of $\operatorname{Mor}_l^{\mathrm{m}}(\mathcal{E})$ where all arrows map to split monics,
- $\operatorname{Mor}_l^{\mathrm{e}}(\mathcal{E})$ the subcategory of $\operatorname{Mor}_l(\mathcal{E})$ where all arrows map to admissible epics and by
- $\operatorname{Mor}_l^{\mathrm{se}}(\mathcal{E})$ the subcategory of $\operatorname{Mor}_l^{\mathrm{e}}(\mathcal{E})$ where all arrows map to split epics.

Remark 1.68 ([BM24, p. 9]). The categories $\operatorname{Mor}_l^{\mathrm{m}}(\mathcal{E})^{\mathrm{op}}$ and $\operatorname{Mor}_l^{\mathrm{e}}(\mathcal{E}^{\mathrm{op}})$ over an exact category $\mathcal{E}$ agree, see Remark 1.5. Therefore, each statement about $\operatorname{Mor}_l^{\mathrm{m}}(\mathcal{E})$ has a dual for $\operatorname{Mor}_l^{\mathrm{e}}(\mathcal{E})$.

Notation 1.69. Let $\mathcal{E}$ be an exact category, $l \in \mathbb{N}$ with $l < N$, and $n \in \mathbb{Z}$. Any object $X^1 \rightarrowtail \cdots \rightarrowtail X^{l+1}$ of $\operatorname{Mor}_l^{\mathrm{m}}(\mathcal{E})$ can be considered as a bounded complex with X^{l+1} at position n and zero otherwise, see Notation 1.20. Homotopies between two such complexes are zero. This gives rise to fully faithful functors

$$\iota^n = \iota^n_{\mathcal{E}} \colon \operatorname{Mor}_l^{\mathrm{m}}(\mathcal{E}) \longrightarrow C_N(\mathcal{E}) \qquad \text{and} \qquad \iota^n = \iota^n_{\mathcal{E}} \colon \operatorname{Mor}_l^{\mathrm{m}}(\mathcal{E}) \longrightarrow \mathcal{K}_N(\mathcal{E}).$$

Notation 1.70 is analogous to Notation 1.47.

Notation 1.70. For an object A of an exact category $\mathcal{E}$ and $t \in \{1, \ldots, l+1\}$, we define the object

$$\mu_t(A)\colon\ 0 \rightarrowtail \cdots \rightarrowtail 0 \rightarrowtail A^{l-t+2} \overset{\mathrm{id}_A}{\rightarrowtail} \cdots \overset{\mathrm{id}_A}{\rightarrowtail} A^{l+1}$$

of $\mathrm{Mor}^{\mathrm{sm}}_l(\mathcal{E})$ by $A^k := A$ for $k \in \{l-t+2, \dots, l+1\}$.

Theorem 1.71 ([BM24, Props. 3.5, 3.9, 3.11])**.** *Let $\mathcal{E}$ be an exact category.*

(a) The category $\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{E})$ is exact with the termwise exact structure.

(b) We have $\mathrm{Proj}(\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{E})) = \mathrm{Mor}^{\mathrm{sm}}_l(\mathrm{Proj}(\mathcal{E}))$ and $\mathrm{Inj}(\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{E})) = \mathrm{Mor}^{\mathrm{sm}}_l(\mathrm{Inj}(\mathcal{E})) = \mathrm{Mor}^{\mathrm{m}}_l(\mathrm{Inj}(\mathcal{E}))$.

(c) If $\mathcal{E}$ has enough projectives resp. injectives, then so has $\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{E})$.

Corollary 1.72 ([BM24, Thm. 3.12])**.** *If $\mathcal{F}$ is a Frobenius category, then so is $\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})$, and $\mathrm{Proj}(\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{F})) = \mathrm{Mor}^{\mathrm{m}}_l(\mathrm{Proj}(\mathcal{F}))$. In particular, the stable category $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{F})$ is triangulated, see Theorem 1.32.* □

Remark 1.73. In general, $\mathrm{Mor}_l(\mathcal{E})$ is not Frobenius even if $\mathcal{E}$ is so, see [BM24, Prop. 3.10].

Lemma 1.74. *Let $\mathcal{E}'$ be a (fully) exact subcategory of $\mathcal{E}$. Then the same holds for the subcategory $\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{E}')$ of $\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{E})$. If $\mathcal{E}'$ has enough $\mathcal{E}$-projectives, then the canonical functor $\underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{E}') \to \underline{\mathrm{Mor}^{\mathrm{m}}}_l(\mathcal{E})$ is fully faithful.*

Proof. The first statement holds due to the termwise exact structure. Suppose now that $\mathcal{E}'$ has enough $\mathcal{E}$-projectives. Then $\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{E}')$ has enough projectives due to Theorem 1.71.(c). By Lemma 1.36.(a) and Theorem 1.71.(b), we have

$$\begin{aligned}\mathrm{Proj}(\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{E}')) = \mathrm{Mor}^{\mathrm{sm}}_l(\mathrm{Proj}(\mathcal{E}')) &= \mathrm{Mor}^{\mathrm{sm}}_l(\mathrm{Proj}(\mathcal{E}) \cap \mathcal{E}') \\ &= \mathrm{Mor}^{\mathrm{sm}}_l(\mathrm{Proj}(\mathcal{E})) \cap \mathrm{Mor}^{\mathrm{m}}_l(\mathcal{E}') = \mathrm{Proj}(\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{E})) \cap \mathrm{Mor}^{\mathrm{m}}_l(\mathcal{E}').\end{aligned}$$

Then $\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{E}')$ has enough $\mathrm{Mor}^{\mathrm{m}}_l(\mathcal{E})$-projectives, see Remark 1.29, and Lemma 1.36.(a) yields the claim. □

1.6. **Idempotent complete categories.**

Definition 1.75 ([Büh10, Def. 6.1])**.** An **idempotent** $e\colon A \to A$ in an additive category $\mathcal{A}$ is an endomorphism with $e^2 = e$. It is called **split** if there is a biproduct decomposition $A \cong eA \oplus (1-e)A$ of A into objects $eA, (1-e)A \in \mathcal{A}$ such that $e \cong \begin{pmatrix} \mathrm{id}_{eA} & 0 \\ 0 & 0 \end{pmatrix}$ with respect to this decomposition.

The category $\mathcal{A}$ is called **idempotent complete** if every idempotent splits or, equivalently, if every idempotent has a kernel, see [Büh10, Rem. 6.2].

Remark 1.76. Let $s\colon B \to A$ and $r\colon A \to B$ be morphisms in an additive category $\mathcal{A}$ with $rs = \mathrm{id}_B$. If the idempotent $e := sr\colon A \to A$ splits, then s induces an isomorphism $B \cong eA$, whose inverse is r restricted to the direct summand eA of A.

Remark 1.77. The conditions on the morphisms defining a biproduct of sequences over an additive category are termwise.

Lemma 1.78. *Let $a\colon A' \to A$ be a morphism in an additive category $\mathcal{A}$. Let $(e', e)\colon a \to a$ be an idempotent in the morphism category of $\mathcal{A}$. If both e' and e are split, then so is (e', e). That is, there* *are unique morphisms* $a' = (e', e)a\colon e'A' \to eA$ *and* $a'' = (1 - (e', e))a\colon (1 - e')A' \to (1 - e)A$ *forming a biproduct* $a \cong a' \oplus a''$ *given termwise by* $A' \cong e'A' \oplus (1 - e')A'$ *and* $A \cong eA \oplus (1 - e)A$*.*

Proof. Due to Remark 1.77, it suffices to construct a' and a'' to fit in the commutative diagrams

$$\begin{tikzcd}
e'A' \arrow[r, dashed, "a'"] \arrow[d, tail, "j'"] & eA \arrow[d, tail, "j"] \\
A' \arrow[r, "a"] \arrow[d, two heads, "q'"] & A \arrow[d, two heads, "q"] \\
(1-e')A' \arrow[r, dashed, "a''"] & (1-e)A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
e'A' \arrow[r, dashed, "a'"] & eA \\
A' \arrow[r, "a"] \arrow[u, two heads, "r'"] & A \arrow[u, two heads, "r"] \\
(1-e')A' \arrow[r, dashed, "a''"] \arrow[u, tail, "s'"] & (1-e)A, \arrow[u, tail, "s"]
\end{tikzcd}$$

where the vertical morphisms define the respective splittings of e' and e. Using $j'r' = e'$ and $jr = e$ we compute $aj'r' = ae' = ea = jra$. Then $qaj'r' = qjra = 0$ as $qj = 0$, and hence $qaj' = 0$ since r' is an epic. As j is a kernel of q and q' a cokernel of j', the two dashed morphisms in the left diagram exist. Then $ja'r' = aj'r' = jra$ and hence $a'r' = ra$ since j is a monic. So, the upper right square commutes, and analogously the lower one. □

Remark 1.79. If (e', e) is an idempotent of a morphism, then e' induces an idempotent of the kernel and e of the cokernel, if the respective object exists.

Lemma 1.80. *Consider a short exact sequence* $A' \overset{i}{\rightarrowtail} A \overset{p}{\twoheadrightarrow} A''$ *in an exact idempotent complete category* $\mathcal{E}$ *and an idempotent* $(e', e)\colon i \to i$ *in the morphism category of* $\mathcal{E}$*. Then there is an idempotent* $(e, e'')\colon p \to p$*. The corresponding morphisms from Lemma 1.78 form a biproduct* $(i, p) \cong (i', p') \oplus (i'', p'')$ *of short exact sequences displayed as follows:*

$$\begin{tikzcd}
e'A' \arrow[r, dashed, tail, "i'"] \arrow[d, tail, "j'"] & eA \arrow[r, dashed, two heads, "p'"] \arrow[d, tail, "j"] & e''A'' \arrow[d, dashed, tail, "j''"] \\
A' \arrow[r, tail, "i"] \arrow[d, two heads, "q'"] & A \arrow[r, two heads, "p"] \arrow[d, two heads, "q"] & A'' \arrow[d, dashed, two heads, "q''"] \\
(1-e')A' \arrow[r, dashed, tail, "i''"] & (1-e)A \arrow[r, dashed, two heads, "p''"] & (1-e'')A''
\end{tikzcd}$$

In particular, applying idempotents preserves admissible monics.

Proof. Clearly, $e''\colon A'' \to A''$ can be chosen as the induced morphism between cokernels, see Remark 1.79. Fix a choice of splittings of the idempotents e', e and e''. Lemma 1.78, applied to the idempotent (e', e) of i and the idempotent (e, e'') of p, yields the dashed morphisms and the desired biproduct, see Remark 1.77. The sequences (i', p') and (i'', p'') are short exact by Corollary 1.11. □

Proposition 1.81.

(a) If $\mathcal{A}$ is an idempotent complete category, then so are $C(\mathcal{A})$ and $C_N(\mathcal{A})$.
(b) If $\mathcal{E}$ is an exact idempotent complete category, then so is $\mathrm{Mor}_l^{\mathrm{m}}(\mathcal{E})$.

Proof.

(a) This follows from Lemma 1.78 and Remarks 1.45 and 1.77.
(b) This follows from Lemma 1.80. □

1.7. **Semiorthogonal decompositions.** In this subsection, we review definitions and results on the topic of *semiorthogonal decompositions* of triangulated categories.

Notation 1.82. Consider a triangulated category $\mathcal{T}$ with suspension functor Σ and subcategories $\mathcal{U}, \mathcal{V}$. Then $\mathcal{U} * \mathcal{V}$ denotes the subcategory of $\mathcal{T}$ consisting of objects $T \in \mathcal{T}$ which fit into a distinguished triangle $U \to T \to V \to \Sigma U$ where $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Definition 1.83. A pair $(\mathcal{U}, \mathcal{V})$ of triangulated subcategories of a triangulated category $\mathcal{T}$ is called a **semiorthogonal decomposition** of $\mathcal{T}$ if

(a) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$ and
(b) $\mathcal{T} = \mathcal{U} * \mathcal{V}$.

Proposition 1.84 ([IKM11, Prop. 1.2])**.** *Let $(\mathcal{U}, \mathcal{V})$ be a semiorthogonal decomposition of a triangulated category $\mathcal{T}$. Then the inclusion functors $i^*\colon \mathcal{U} \to \mathcal{T}$ and $j_*\colon \mathcal{V} \to \mathcal{T}$ have a respective right adjoint $i^!\colon \mathcal{T} \to \mathcal{U}$ and left adjoint $j^*\colon \mathcal{T} \to \mathcal{V}$. These induce triangle equivalences $\mathcal{T}/\mathcal{V} \simeq \mathcal{U}$ and $\mathcal{T}/\mathcal{U} \simeq \mathcal{V}$.* □

Remark 1.85. Let $\mathcal{T}$ be a triangulated category with suspension functor Σ. The covariant (contravariant) Hom-functor is (co)homological: Given a distinguished triangle

$$Y \xrightarrow{f} Y' \xrightarrow{g} Y'' \longrightarrow \Sigma Y,$$

each of the following two solid commutative diagrams can be completed by a dashed arrow:

$$\begin{tikzcd}
& X \arrow[dl, dashed] \arrow[d, "f'"'] \arrow[dr, "0"] & \\
Y \arrow[r, "f"] & Y' \arrow[r, "g"] & Y''
\end{tikzcd}
\qquad
\begin{tikzcd}
& X & \\
Y \arrow[ur, "0"] \arrow[r, "f"] & Y' \arrow[u, "g'"'] \arrow[r, "g"] & Y'' \arrow[ul, dashed]
\end{tikzcd}$$

Remark 1.86. The adjoints in Proposition 1.84 can be made explicit as follows, see [Bon90, Proof of Lem. 3.1.(b) ⇒ (c)]:

- For $T \in \mathcal{T} = \mathcal{U} * \mathcal{V}$, there is a distinguished triangle in $\mathcal{T}$ which defines $i^!T \in \mathcal{U}$ up to isomorphism:

$$i^!T \xrightarrow{u_T} T \longrightarrow V_T \longrightarrow \Sigma i^!T$$

- Let $f\colon T \to T'$ be a morphism in $\mathcal{T}$. Using $\mathrm{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$ and Remark 1.85, $i^!f$ is uniquely determined by the commutative diagram

$$\begin{tikzcd}
& i^! T \arrow[ddl, dashed, "i^! f"'] \arrow[d, "u_T"] \arrow[ddr, "0"] & \\
& T \arrow[d, "f"] & \\
i^! T' \arrow[r, "u_{T'}"] & T' \arrow[r] & V_{T'}.
\end{tikzcd}$$

The functor j^* is given dually.

Theorem 1.87 ([JK15, Lem. 1.1, Thm. B])**.** *Consider triangulated subcategories $\mathcal{U}, \mathcal{V}$ of a triangulated category $\mathcal{T}$. Then the following conditions are equivalent:*

1. *$\mathcal{V} * \mathcal{U} \subseteq \mathcal{U} * \mathcal{V}$.*
2. *$\mathcal{U} * \mathcal{V}$ is a triangulated subcategory of $\mathcal{T}$.*
3. *Any morphism in $\operatorname{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V})$ factors through an object of $\mathcal{U} \cap \mathcal{V}$.*

In this case, the following statements hold:

- (a) *The pair $(\mathcal{U}/(\mathcal{U} \cap \mathcal{V}), \mathcal{V}/(\mathcal{U} \cap \mathcal{V}))$ is a semiorthogonal decomposition of $(\mathcal{U} * \mathcal{V})/(\mathcal{U} \cap \mathcal{V})$.*
- (b) *There are triangle equivalences $\mathcal{U}/(\mathcal{U} \cap \mathcal{V}) \simeq (\mathcal{U} * \mathcal{V})/\mathcal{V}$ and $\mathcal{V}/(\mathcal{U} \cap \mathcal{V}) \simeq (\mathcal{U} * \mathcal{V})/\mathcal{U}$.*
- (c) *The canonical functors $\mathcal{U}/(\mathcal{U} \cap \mathcal{V}) \to \mathcal{T}/\mathcal{V}$ and $\mathcal{V}/(\mathcal{U} \cap \mathcal{V}) \to \mathcal{T}/\mathcal{U}$ are fully faithful.* □

2. N-ACYCLICITY

In this section, we define (total) N-acyclicity of sequences over an exact category $\mathcal{E}$ (Subsection 2.1). We show that it is preserved locally under extensions, cones and suspensions (Subsection 2.2). This leads to various triangulated subcategories of $\mathcal{K}_N(\mathcal{E})$ known from the classical case. We describe acyclic N-complexes over $\mathcal{E}$ equivalently in terms of so-called *acyclic N-arrays* (Subsection 2.3). To resolve bounded above N-complexes over $\mathcal{E}$, we extend a construction of Keller from the case $N = 2$ (Subsection 2.4). If $\mathcal{E}$ is Frobenius, we relate complete resolutions with their N-syzygies (Subsection 2.5).

2.1. **Contraction and acyclicity.**

In this subsection, we define the concept of N-acyclicity by reduction to the special case $N = 2$, see Definition 1.40.

Definition 2.1. Let $\mathcal{A}$ be an additive category. Given $N \in \mathbb{N}_{\geq 2}$, $n \in \mathbb{Z}$, and $r \in \{1, \dots, N-1\}$, the **contraction functor** $\gamma^n_{r,N} \colon C(\mathcal{A}) \to C(\mathcal{A})$ is defined by sending $X \in C(\mathcal{A})$ to the sequence

$$\cdots \longrightarrow X^{n-N} \xrightarrow{d_X^{\{r\}}} X^{n-N+r} \xrightarrow{d_X^{\{N-r\}}} X^n \xrightarrow{d_X^{\{r\}}} X^{n+r} \xrightarrow{d_X^{\{N-r\}}} X^{n+N} \xrightarrow{d_X^{\{r\}}} X^{n+N+r} \longrightarrow \cdots$$

with X^n at position n, see Notation 1.57. A morphism $(f^k)_{k \in \mathbb{Z}}$ in $C(\mathcal{A})$ is mapped to

$$(\dots, f^{n-N+r}, f^n, f^{n+r}, f^{n+N}, \dots).$$

By definition, $\gamma^n_{r,N}$ restricts to a functor $\gamma^n_r \colon C_N(\mathcal{A}) \to C_2(\mathcal{A})$. If $\mathcal{E}$ is an exact category, $\gamma^n_{r,N} \colon C(\mathcal{E}) \to C(\mathcal{E})$ is exact with respect to the termwise exact structure, see Remark 1.22.

Definition 2.2. Let $\mathcal{E}$ be an exact category and $N \in \mathbb{N}_{\geq 2}$. We call a sequence $X \in \mathcal{C}(\mathcal{E})$ ***N*-acyclic at position** $n \in \mathbb{Z}$ if $\gamma_r^n(X)$ is 2-acyclic at position n for all $r \in \{1, \dots, N-1\}$, see Definition 1.40.(b). We call it **totally *N*-acyclic at position** $n \in \mathbb{Z}$ if, in addition, $\operatorname{Hom}_{\mathcal{E}}(X, P) \in \mathcal{C}_N(\mathrm{Ab})$ is N-acyclic at position $-n$ for all $P \in \operatorname{Proj}(\mathcal{E})$, see Notation 1.46. We say that X is **(totally) *N*-acyclic** if it is (totally) N-acyclic at all positions $n \in \mathbb{Z}$.
The subcategory of $\mathcal{C}(\mathcal{E})$ consisting of N-acyclic sequences is denoted by $\mathcal{C}_N^{\infty,\varnothing}(\mathcal{E})$, its subcategory of totally N-acyclic sequences by $\mathcal{C}_N^{\infty,\varnothing^*}(\mathcal{E})$. This notation becomes clear in Notation 2.19. Note that $\mathcal{C}_N^{\infty,\varnothing}(\mathcal{E})$ is a subcategory of $\mathcal{C}_N(\mathcal{E})$. For this reason we also use the term **(totally) acyclic *N*-complex**.

Remark 2.3. If $\mathcal{F}$ is a Frobenius category, then $\mathcal{C}_N^{\infty,\varnothing}(\mathcal{F}) = \mathcal{C}_N^{\infty,\varnothing^*}(\mathcal{F})$ as any $P \in \operatorname{Proj}(\mathcal{F}) = \operatorname{Inj}(\mathcal{F})$ makes $\operatorname{Hom}_{\mathcal{F}}(-, P)$ an exact functor.

Remark 2.4.

(a) Over an additive category $\mathcal{A}$, we have $\operatorname{Hom}_{\mathcal{A}}(A, -) \circ \gamma_r^n = \gamma_r^n \circ \operatorname{Hom}_{\mathcal{A}}(A, -)$ and $\operatorname{Hom}_{\mathcal{A}}(-, A) \circ \gamma_r^n = \gamma_{N-r}^{-n} \circ \operatorname{Hom}_{\mathcal{A}}(-, A)$ as functors on $\mathcal{C}(\mathcal{A})$, for all $A \in \mathcal{A}$, $n \in \mathbb{Z}$, and $r \in \{1, \dots, N-1\}$.

(b) In particular, an N-complex $X \in \mathcal{C}_N(\mathcal{E})$ over an exact category $\mathcal{E}$ is totally N-acyclic at position $n \in \mathbb{Z}$ if and only if $\gamma_r^n(X)$ is totally 2-acyclic at n, for all $r \in \{1, \dots, N-1\}$.

Remark 2.5. Let $\mathcal{E}$ be an exact category, $n \in \mathbb{Z}$, and $r \in \{1, \dots, N-1\}$. Note that $Z_{(r)}^n(X) = Z^n(\gamma_r^n(X))$ and $B_{(r)}^n(X) = B^n(\gamma_{N-r}^n(X))$ for any sequence $X \in \mathcal{C}(\mathcal{E})$ if the respective objects exist, see Notation 1.61. In this case, X is N-acyclic at position n if and only if $Z_{(r)}^n(X) = B_{(N-r)}^n(X)$ for all r. In particular, $H_{(r)}^n = H^n \circ \gamma_r^n$ if $\mathcal{E}$ is Abelian and $X \in \mathcal{C}_N(\mathcal{E})$ an N-complex. In this case, X is N-acyclic at position n if and only if $H_{(r)}^n(X) = 0$ for all r.

Lemma 2.6. *Consider a term*wise *short exact sequence* $X \rightarrowtail Y \twoheadrightarrow Z$ *of N-complexes over an exact category* $\mathcal{E}$*. If X, Y, and Z are acyclic at position* $n \in \mathbb{Z}$*, then the induced sequence*

$$Z_{(r)}^n(X) \rightarrowtail Z_{(r)}^n(Y) \twoheadrightarrow Z_{(r)}^n(Z)$$

is short exact for all $r \in \{1, \dots, N-1\}$*. The verbatim statement holds for cokernels instead of kernels.*

Proof. Due to Remark 2.5, we may assume that $N = 2$. There is a commutative diagram

$$\begin{tikzcd}
X^{n-1} \arrow[r, two heads] \arrow[d, tail] & Z^n(X) \arrow[r, tail, "x'"] \arrow[d, dashed, "z'"', "y'"] & X^n \arrow[r, "\circ" marking] \arrow[d, tail, "v'"] \arrow[dr, "x"] & X^{n+1} \arrow[d, tail] \\
Y^{n-1} \arrow[r, two heads, "p"] \arrow[d, two heads] & Z^n(Y) \arrow[r, tail, "w'"] \arrow[d, dashed, "z"'] \arrow[dr, "y"] & Y^n \arrow[r, "w" marking] \arrow[d, two heads, "v"] & Y^{n+1} \arrow[d, two heads] \\
Z^{n-1} \arrow[r, two heads] & Z^n(Z) \arrow[r, tail] & Z^n \arrow[r, "\circ" marking] & Z^{n+1},
\end{tikzcd}$$

where z and z' are induced on the respective kernels, see Remark 2.7.(b). By Lemma 2.8, z' is a kernel of z. Since zp is an admissible epic, so is z due to Proposition 1.10.(b). It follows that

$$\begin{tikzcd}
Z^n(X) \arrow[r, tail, "z'"] & Z^n(Y) \arrow[r, two heads, "z"] & Z^n(Z)
\end{tikzcd}$$

is a short exact sequence in $\mathcal{E}$ as desired. □

Remark 2.7. Consider a not necessarily commutative diagram

$$\begin{tikzcd}
\bullet \arrow[r, "e"] \arrow[d] \arrow[dr, phantom, "(X)"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "(Y)"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r, "m"] & \bullet
\end{tikzcd}$$

in an arbitrary category, and suppose that the outer rectangle (XY) commutes. Then:

(a) If e is an epic and (X) commutes, then (Y) commutes as well.
(b) If m is a monic and (Y) commutes, then (X) commutes as well.

Lemma 2.8. *In a category with zero morphisms, consider a commutative diagram*

$$\begin{tikzcd}
& \bullet \arrow[dl, equal] \arrow[dr, "x'"] & & \\
\bullet \arrow[dr, "y'"] & & \bullet \arrow[d, "v'"] \arrow[dr, "x"] & \\
& \bullet \arrow[r, hook, "w'"] \arrow[dr, "y"] & \bullet \arrow[r, "w"] \arrow[d, "v"] & \bullet \\
& & \bullet &
\end{tikzcd}$$

with $ww' = 0$ *and* x' *and* v' *kernels of* x *and* v*, respectively. Then* y' *is a kernel of* y*.*

Proof. First note that $yy' = vv'x' = 0$ since v' is a kernel of v. To prove universality, let a be a morphism with $ya = 0$. Then $0 = vw'a$, and hence $w'a = v'b$ for a unique b since v' is a kernel of v. We compute $xb = wv'b = ww'a = 0$ using $ww' = 0$. Hence, $b = x'c$ for a unique c since x' is a kernel of x.

$$\begin{tikzcd}
& \bullet \arrow[dl, equal] \arrow[dr, "x'"] & & \\
\bullet \arrow[dr, "y'"] & \bullet \arrow[l, dashed, "c"] \arrow[u, dashed, "c"] \arrow[r, dashed, "b"] \arrow[d, "a"] & \bullet \arrow[d, "v'"] \arrow[dr, "x"] & \\
& \bullet \arrow[r, hook, "w'"] \arrow[dr, "y"] & \bullet \arrow[r, "w"] \arrow[d, "v"] & \bullet \\
& & \bullet &
\end{tikzcd}$$

We show that c is the unique morphism with $y'c = a$ as required: Note that $w'y'c = v'x'c = v'b = w'a$, which implies $y'c = a$ since w' is monic. If c' is an arbitrary morphism with $y'c' = a$, then $v'x'c' = w'y'c' = w'a$. Thus, $x'c' = b$, and hence $c' = c$ by uniqueness of b and c, respectively. □

Notation 2.9. For $N \in \mathbb{N}_{\geq 2}$, $n \in \mathbb{Z}$ and, $r \in \{1, \ldots, N-1\}$, consider the set

$$\mathcal{Z} = \mathcal{Z}_r^n := \{n + aN + br \,|\, a \in \mathbb{Z}, b \in \{0, 1\}\} \subseteq \mathbb{Z}$$

of indices selected by the contraction functor $\gamma := \gamma_r^n$. Note that for any $s \in \mathbb{Z}$ the interval $[s-N+1, s]$ contains exactly two elements of $\mathcal{Z}$. Denote by $\gamma' \colon \mathcal{Z} \to \mathbb{Z}$, $n + aN + br \mapsto n + 2a + b$, the reindexing realized by γ. By abuse of notation, we also write

$$\gamma = \gamma_r^n \colon \mathbb{Z} \longrightarrow \mathbb{Z},\ s \longmapsto \gamma'(\max\{k \in \mathcal{Z} \,|\, k \leq s\}),$$

which extends γ' to $\mathbb{Z}$.

Remark 2.10. Let $\mathcal{E}$ be an exact category, $A \in \mathcal{E}$, and $s \in \mathbb{Z}$.

(a) Using Notation 2.9, we have $\gamma_r^n(\mu_N^s(A)) = \mu_2^{\gamma_r^n(s)}(A)$ for all $n \in \mathbb{Z}$ and $r \in \{1, \ldots, N-1\}$.

(b) Suppose that $A \neq 0$. Then the sequence $\mu_t^s(A)$ is (totally) N-acyclic if and only if $t = N$: Indeed, $\gamma_r^n(\mu_N^s(A))$ is 2-acyclic for all $n \in \mathbb{Z}$ and $r \in \{1, \ldots, N-1\}$ due to (a). Then also $\mathrm{Hom}_{\mathcal{E}}(\mu_N^s(A), P) = \mu_N^{-s+N-1}(\mathrm{Hom}_{\mathcal{E}}(A, P))$ is N-acyclic for any $P \in \mathrm{Proj}(\mathcal{E})$, see Remark 1.48. Thus, $\mu_N^s(A)$ is even totally N-acyclic. If $t \in \{1, \ldots, N-1\}$, then the 2-complex $\gamma_1^s(\mu_t^s(A)) = \mu_1^s(A)$ is clearly not 2-acyclic. So, $\mu_N^s(A)$ is not N-acyclic.

2.2. **Cones and extensions.** In this subsection, we describe to which extent the cone $C(f)$ and the cocone $C^*(f)$, see Construction 1.31, preserve local N-acyclicity of the source and target of f (Proposition 2.17). This specializes to the suspension functor Σ and its quasi-inverse Σ^{-1} (Corollary 2.18). We examine the preservation of local N-acyclicity is preserved under extensions (Proposition 2.13). Imposing boundedness and acyclicity conditions, defines various extension-closed Frobenius subcategories of $C_N(\mathcal{E})$. Their stable categories are related in a diagram of triangulated subcategories of $\mathcal{K}_N(\mathcal{E})$ (Theorem 2.20).

Lemma 2.11 ([Büh10, Lem. 10.3])**.** *Let $f \colon X \to Y$ be a morphism of 2-complexes over an exact category $\mathcal{E}$. If X and Y are acyclic at positions $n, n+1 \in \mathbb{Z}$, then the cone $C(f)$ of f is acyclic at position n. Dually, if X and Y are acyclic at positions $n-1, n \in \mathbb{Z}$, then the cocone $C^*(f)$ of f is acyclic at position n.*

Proof. Although Bühler's Lemma concerns global acyclicity, his arguments prove our local claim. The dual statement then follows by Construction 1.58.(c). □

Remark 2.12. If a sequence $X \in C(\mathcal{E})$ over an exact category $\mathcal{E}$ is N-acyclic at positions $n, n+1, \ldots, n+N-1 \in \mathbb{Z}$, resp. $n-N+1, \ldots, n-1, n \in \mathbb{Z}$, then $\gamma_r^n(X)$ is 2-acyclic at positions $n, n+1$, resp. $n-1, n$, for all $r \in \{1, \ldots, N-1\}$. The verbatim statement for total N-acyclicity follows due to Remark 2.4.(b).

Proposition 2.13. *Let $X \rightarrowtail Y \twoheadrightarrow Z$ be a termsplit short exact sequence of N-complexes $X, Y, Z \in C_N(\mathcal{E})$ over an exact category $\mathcal{E}$.*

(a) If X is acyclic at positions $n, n+1, \ldots, n+N-1 \in \mathbb{Z}$ and and Z is acyclic at positions $n-N+1, \ldots, n-1, n$, then Y is acyclic at position n.

(b) If X and Z are totally acyclic at positions $n-N+1,\dots,n,\dots,n+N-1\in\mathbb{Z}$, then Y is totally acyclic at position n.

In particular, the subcategories $C_N^{\infty,\varnothing}(\mathcal{E})$ and $C_N^{\infty,\varnothing^*}(\mathcal{E})$ of (totally) acyclic N-complexes are both extension-closed in $C_N(\mathcal{E})$.

Proof.

(a) By Remark 2.12, the claim reduces to the case $N=2$. By hypothesis, there are decompositions $Y^k\cong X^k\oplus Z^k$ for all k. We can thus write the kth morphism in Y as

$$d_Y^k=\begin{pmatrix} d_X^k & f^k\\ 0 & d_Z^k\end{pmatrix}$$

for some morphism $f^k\colon Z^k\to X^{k+1}$ in $\mathcal{E}$. Since Y is a complex, we have

$$0=d_Y^2=\begin{pmatrix} d_X^2 & d_Xf+fd_Z\\ 0 & d_Z^2\end{pmatrix}.$$

Consequently, $f\circ(-d_Z)=d_X\circ f$ and $\Theta^{-1}f\colon -\Theta^{-1}Z\to X$ is a morphism in $C_N(\mathcal{E})$ with

$$d^k_{C(\Theta^{-1}f)}=\begin{pmatrix} d_X^k & (\Theta^{-1}f)^{k+1}\\ 0 & -d^{k+1}_{-\Theta^{-1}Z}\end{pmatrix}=\begin{pmatrix} d_X^k & f^k\\ 0 & d_Z^k\end{pmatrix}=d_Y^k.$$

Since $-\Theta^{-1}Z$ and X are acyclic at positions $n,n+1$ by hypothesis, $Y\cong C(\Theta^{-1}f)$ is acyclic at position n due to Lemma 2.11.

(b) This follows from (a) applied to the termsplit short exact sequence

$$\operatorname{Hom}_{\mathcal{E}}(Z,P)\rightarrowtail\operatorname{Hom}_{\mathcal{E}}(Y,P)\twoheadrightarrow\operatorname{Hom}_{\mathcal{E}}(X,P)$$

for each $P\in\operatorname{Proj}(\mathcal{E})$. □

Lemma 2.14. *If $A'\overset{a'}{\rightarrowtail}A\overset{a}{\twoheadrightarrow}A''$ is a (split) short exact sequence in an exact category $\mathcal{E}$, then so are the following sequences:*

(a) $A'\oplus B\xrightarrow{\begin{pmatrix} a' & b\\ 0 & \mathrm{id}_B\end{pmatrix}}A\oplus B\xrightarrow{\begin{pmatrix} a & -ab\end{pmatrix}}A''$ *for any morphism $b\colon B\to A$ in $\mathcal{E}$, and*

(b) $A'\xrightarrow{\begin{pmatrix} a'\\ -ca'\end{pmatrix}}A\oplus C\xrightarrow{\begin{pmatrix} a & 0\\ c & \mathrm{id}_C\end{pmatrix}}A''\oplus C$ *for any morphism $c\colon A\to C$ in $\mathcal{E}$.*

Proof. The following isomorphisms of short exact sequences in $\mathcal{E}$ prove the claims:

(a)

$$\begin{tikzcd}[column sep=huge, row sep=huge]
A' \oplus B \arrow[r, "{\begin{pmatrix} a' & b \\ 0 & \mathrm{id}_B \end{pmatrix}}"] \arrow[d, "\mathrm{id}_{A'\oplus B}"] & A \oplus B \arrow[r, "{\begin{pmatrix} a & -ab \end{pmatrix}}"] \arrow[d, "\cong"', "{\begin{pmatrix} \mathrm{id}_A & -b \\ 0 & \mathrm{id}_B \end{pmatrix}}"] & A'' \arrow[d, "\mathrm{id}_{A''}"] \\
A' \oplus B \arrow[r, "{\begin{pmatrix} a' & 0 \\ 0 & \mathrm{id}_B \end{pmatrix}}"'] & A \oplus B \arrow[r, "{\begin{pmatrix} a & 0 \end{pmatrix}}"'] & A''
\end{tikzcd}$$

(b)

$$\begin{tikzcd}[column sep=huge, row sep=huge]
A' \arrow[r, "{\begin{pmatrix} a' \\ -ca' \end{pmatrix}}"] \arrow[d, "\mathrm{id}_{A'}"] & A \oplus C \arrow[r, "{\begin{pmatrix} a & 0 \\ c & \mathrm{id}_C \end{pmatrix}}"] \arrow[d, "\cong"', "{\begin{pmatrix} \mathrm{id}_A & 0 \\ c & \mathrm{id}_C \end{pmatrix}}"] & A'' \oplus C \arrow[d, "\mathrm{id}_{A''\oplus C}"] \\
A' \arrow[r, "{\begin{pmatrix} a' \\ 0 \end{pmatrix}}"'] & A \oplus C \arrow[r, "{\begin{pmatrix} a & 0 \\ 0 & \mathrm{id}_C \end{pmatrix}}"'] & A'' \oplus C
\end{tikzcd}$$

□

Construction 2.15. Let $f\colon X \to Y$ be a morphism of N-complexes over an exact category $\mathcal{E}$. Using $\gamma = \gamma_r^n$ and $\mathcal{Z} = \mathcal{Z}_r^n$ from Notation 2.9, we have, see Construction 1.51 and Remark 2.10.(a),

$$\gamma I_N(X) = \bigoplus_{k\in\mathbb{Z}} \gamma(\mu_N^k(X^k)) = \bigoplus_{k\in\mathbb{Z}} \mu_2^{\gamma(k)}(X^k),$$

which is a direct sum of $I_2(\gamma X) = \bigoplus_{k\in\mathcal{Z}} \mu_2^{\gamma(k)}(X^k)$ and $C = \bigoplus_{k\in\mathbb{Z}\setminus\mathcal{Z}} \mu_2^{\gamma(k)}(X^k)$. This leads to a split short exact sequence

$$\begin{tikzcd}[column sep=large]
I_2(\gamma X) \arrow[r, tail, "j_X"] & \gamma I_N(X) \arrow[r, two heads, "q_X"] & C
\end{tikzcd}$$

with $C \in \mathrm{Proj}(\mathcal{C}_N(\mathcal{E})) \cap C_N^{\infty,\varnothing}(\mathcal{E})$, see Lemma 1.50 and Remark 2.10.(b). This persists after twisting the maps by $D^k := \begin{pmatrix} d_X \\ \vdots \\ d_X^{\{k\}} \end{pmatrix}$ as follows:

$$j_X = \begin{pmatrix} 1 & 0 \\ D^{r-1} & 0 \\ 0 & 1 \\ 0 & D^{N-r-1} \end{pmatrix} \quad \text{and} \quad q_X = \begin{pmatrix} -D^{r-1} & E_{r-1} & 0 & \\ 0 & 0 & -D^{N-r-1} & E_{N-r-1} \end{pmatrix}.$$

Indeed, permuting the direct summands yields

$$(j_X, p_X) \cong \left(\begin{pmatrix} D \\ E_2 \end{pmatrix}, \begin{pmatrix} E_{N-2} & -D \end{pmatrix}\right), \text{ where } D := \begin{pmatrix} D^{r-1} & 0 \\ 0 & D^{N-r-1} \end{pmatrix}.$$

This is a split short exact sequence due to Lemma 2.14.(a) applied to $a' = 0$, $b = D$ and $a = E_{N-2}$. With $i_{\gamma X} = \begin{pmatrix} 1 \\ d_X^{\{r\}} \end{pmatrix}$ and $\gamma i_X = \begin{pmatrix} 1 \\ D^{N-1} \end{pmatrix}$, see Constructions 1.31 and 1.51, we obtain

$$j_X \circ i_{\gamma X} = \gamma i_X.$$

We now apply the exact functor γ to the sequence $(S(f))$ and relate the result with $(S(\gamma f))$, see Constructions 1.31 and 1.51. By Lemma 1.15, the commutative square given by the preceding formula can be extended to the following commutative diagram in $C_N(\mathcal{E})$ with termsplit short exact rows and split columns:

$$\begin{tikzcd}
\gamma X \arrow[r, tail, "\begin{pmatrix} i_{\gamma X} \\ -\gamma f \end{pmatrix}"] \arrow[d, tail, "\mathrm{id}"'] & I_2(\gamma X) \oplus \gamma Y \arrow[r, two heads] \arrow[d, tail, "\begin{pmatrix} j_X & 0 \\ 0 & \mathrm{id}_{\gamma Y} \end{pmatrix}"] & C(\gamma f) \arrow[d, tail, dashed, "c_X"] \\
\gamma X \arrow[r, tail, "\begin{pmatrix} \gamma i_X \\ \gamma(-f) \end{pmatrix}"] \arrow[d, two heads] & \gamma I_N(X) \oplus \gamma Y \arrow[r, two heads] \arrow[d, two heads, "\begin{pmatrix} q_X & 0 \end{pmatrix}"] & \gamma C(f) \arrow[d, two heads, dashed] \\
0 \arrow[r, tail] & C \arrow[r, two heads, "\mathrm{id}"] & C
\end{tikzcd} \tag{2.1}$$

We summarize this result in the following

Lemma 2.16. *Let $f\colon X \to Y$ a morphism of N-complexes over an exact category $\mathcal{E}$. Consider the contraction functor γ_r^n, for some $n \in \mathbb{Z}$ and $r \in \{1, \ldots, N-1\}$. Then there is a split admissible monic $c_X\colon C(\gamma_r^n(f)) \rightarrowtail \gamma_r^n(C(f))$ with cokernel in $\operatorname{Proj}(C_N(\mathcal{E})) \cap C_N^{\infty,\varnothing^*}(\mathcal{E})$. In particular:*

(a) The morphism c_X is an isomorphism in $\mathcal{K}_2(\mathcal{E})$.

(b) If $C(\gamma_r^n(f))$ is (totally) 2-acyclic at position n, then so is $\gamma_r^n(C(f))$. □

Proposition 2.17. *Let $f\colon X \to Y$ be a morphism of N-complexes over an exact category $\mathcal{E}$. If X and Y are (totally) N-acyclic at positions $n, n+1, \ldots, n+N-1 \in \mathbb{Z}$, then the cone $C(f)$ of f is (totally) N-acyclic at position n. Dually, if X and Y are (totally) N-acyclic at positions $n-N+1, \ldots, n-1, n \in \mathbb{Z}$, then the cocone $C^*(f)$ of f is (totally) N-acyclic at position n.*

Proof. By Remark 2.12, $\gamma_r^n(X)$ and $\gamma_r^n(Y)$ are (totally) 2-acyclic at positions n, $n+1$ for all $r \in \{1, \ldots, N-1\}$. Lemmas 2.11 and 2.16.(b) then yield 2-acyclicity of $C(\gamma_r^n(f))$, and hence of $\gamma_r^n(C(f))$ at position n for all r. This means exactly that $C(f)$ is N-acyclic at position n. Assuming total acyclicity, $\mathrm{Hom}_{\mathcal{E}}(\gamma_r^n(X), P)$ and $\mathrm{Hom}_{\mathcal{E}}(\gamma_r^n(Y), P)$ are 2-acyclic at positions $-n-1, -n$ for all r and $P \in \mathrm{Proj}(\mathcal{E})$. Using Remark 1.53.(b), $\mathrm{Hom}_{\mathcal{A}}(C(\gamma_r^n(f)), P) \cong C^*(\mathrm{Hom}_{\mathcal{A}}(\gamma_r^n(f), P))$ is then 2-acyclic at position $-n$ by Lemma 2.11. This proves total 2-acyclicity of $C(\gamma_r^n(f))$, and hence of $\gamma_r^n(C(f))$ at n for all r due to Lemma 2.16.(b). By Remark 2.4.(b), $C(f)$ is now totally N-acyclic at position n. ☐

In view of Construction 1.58.(b), we obtain, as a special case, the following

Corollary 2.18. *Let $X \in C_N(\mathcal{E})$ be an N-complex over an exact category $\mathcal{E}$. If X is (totally) acyclic at positions $n, n+1, \dots, n+N-1 \in \mathbb{Z}$, then ΣX is (totally) acyclic at position n. Dually, if X is (totally) acyclic at positions $n-N+1, \dots, n-1, n \in \mathbb{Z}$, then $\Sigma^{-1} X$ is (totally) acyclic at position n.* ☐

Notation 2.19. We use Verdier's notation for various subcategories of N-complexes:

- For an additive category $\mathcal{A}$, let $C_N^-(\mathcal{A})$, $C_N^+(\mathcal{A})$ and $C_N^b(\mathcal{A})$ denote the subcategories of $C_N(\mathcal{A})$ consisting of N-complexes $X \in C_N(\mathcal{A})$ with $X^k = 0$ for any sufficiently large, any sufficiently small, and almost all $k \in \mathbb{Z}$, respectively. These subcategories are extension-closed and hence fully exact in $C_N(\mathcal{A})$, see Lemma 1.19.(a). To afford the following notation, we set $C_N^\infty(\mathcal{A}) := C_N(\mathcal{A})$. For $\# \in \{\infty, +, -, b\}$, the projectively stable category of $C_N^\#(\mathcal{A})$ is denoted by $\mathcal{K}_N^\#(\mathcal{A})$.
- For an exact category $\mathcal{E}$ and $\# \in \{\infty, +, -, b\}$, let $C_N^{\#,-}(\mathcal{E})$, $C_N^{\#,+}(\mathcal{E})$, $C_N^{\#,b}(\mathcal{E})$ and $C_N^{\#,\varnothing}(\mathcal{E})$ denote the subcategories of $C_N^\#(\mathcal{E}) =: C_N^{\#,\infty}(\mathcal{E})$ consisting of N-complexes which are acyclic at all sufficiently large, sufficiently small, almost all, and all positions, respectively. In addition, we denote by $C_N^{\#,\varnothing^*}(\mathcal{E})$ the subcategory of $C_N^{\#,\varnothing}(\mathcal{E})$ consisting of totally acyclic N-complexes. For $\natural \in \{\infty, +, -, b, \varnothing, \varnothing^*\}$, the associated projectively stable category is denoted by $\mathcal{K}_N^{\#,\natural}(\mathcal{A})$.
- It convenient to use the obvious partial ordering

$$\begin{tikzcd}[column sep=small, row sep=small]
 & \infty \arrow[dl, dash] \arrow[dr, dash] & \\
+ \arrow[dr, dash] & & - \arrow[dl, dash] \\
 & b \arrow[d, dash] & \\
 & \varnothing \arrow[d, dash] & \\
 & \varnothing^*. &
\end{tikzcd}$$

- Let $\mathcal{A}$ be a subcategory of an exact category $\mathcal{E}$. For $\# \in \{\infty, +, -, b\}$ and $\natural \in \{\infty, +, -, b, \varnothing, \varnothing^*\}$, we abbreviate $C_N^{\#,\natural_{\mathcal{E}}}(\mathcal{A}) := C_N(\mathcal{A}) \cap C_N^{\#,\natural}(\mathcal{E})$, and denote its projectively stable category by $\mathcal{K}_N^{\#,\natural_{\mathcal{E}}}(\mathcal{A})$. There are two notable special cases:
  - $\mathrm{APC}_N(\mathcal{E}) := C^{\infty,\varnothing_{\mathcal{E}}}(\mathrm{Proj}(\mathcal{E}))$, the category of acyclic N-complexes of projectives, and
  - $\mathrm{TAPC}_N(\mathcal{E}) := C^{\infty,\varnothing^*_{\mathcal{E}}}(\mathrm{Proj}(\mathcal{E}))$, the category of totally acyclic N-complexes of projectives.

  They coincide if $\mathcal{E}$ is Frobenius, see Remark 2.3.

Theorem 2.20. *Let $\mathcal{E}$ be an exact category, $\# \in \{\infty, +, -, b\}$, and $\natural \in \{\infty, +, -, b, \varnothing, \varnothing^*\}$.*

(a) The subcategory $C_N^{\#,\natural}(\mathcal{E})$ is extension-closed and hence fully exact in $C_N(\mathcal{E})$.

(b) The category $C_N^{\#,\natural}(\mathcal{E})$ is a sub-Frobenius category of $C_N(\mathcal{E})$.

(c) There are the following diagrams of canonical fully faithful, triangulated functors:

$$\begin{tikzcd}
 & & & \mathcal{K}_N^{+}(\mathcal{E}) \arrow[dr] & & \\
 & \mathcal{K}_N^{+,\varnothing}(\mathcal{E}) \arrow[r] \arrow[dr] & \mathcal{K}_N^{+,b}(\mathcal{E}) \arrow[ur] \arrow[dr] & & \mathcal{K}_N^{\infty,+}(\mathcal{E}) \arrow[dr] & \\
\mathcal{K}_N^{b,\varnothing}(\mathcal{E}) \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{K}_N^{b}(\mathcal{E}) \arrow[ur] \arrow[dr] & \mathcal{K}_N^{\infty,\varnothing}(\mathcal{E}) \arrow[r] & \mathcal{K}_N^{\infty,b}(\mathcal{E}) \arrow[ur] \arrow[dr] & & \mathcal{K}_N(\mathcal{E}) \\
 & \mathcal{K}_N^{-,\varnothing}(\mathcal{E}) \arrow[r] \arrow[ur] & \mathcal{K}_N^{-,b}(\mathcal{E}) \arrow[ur] \arrow[dr] & & \mathcal{K}_N^{\infty,-}(\mathcal{E}) \arrow[ur] & \\
 & & & \mathcal{K}_N^{-}(\mathcal{E}) \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{K}_N^{+,\varnothing^*}(\mathcal{E}) \arrow[r] \arrow[dr] & \mathcal{K}_N^{+,\varnothing}(\mathcal{E}) \arrow[dr] & \\
\mathcal{K}_N^{b,\varnothing^*}(\mathcal{E}) \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{K}_N^{b,\varnothing}(\mathcal{E}) \arrow[ur] \arrow[dr] & \mathcal{K}_N^{\infty,\varnothing^*}(\mathcal{E}) \arrow[r] & \mathcal{K}_N^{\infty,\varnothing}(\mathcal{E}) \\
 & \mathcal{K}_N^{-,\varnothing^*}(\mathcal{E}) \arrow[r] \arrow[ur] & \mathcal{K}_N^{-,\varnothing}(\mathcal{E}) \arrow[ur] &
\end{tikzcd}$$

For a subcategory $\mathcal{A}$ of $\mathcal{E}$, these statements hold more generally with $\mathcal{E}$ replaced by $\mathcal{A}$, and $\natural$ by $\natural_{\mathcal{E}}$.

Proof. Part (a) follows from Lemma 1.19.(a) and Proposition 2.13. For any $X \in C_N^{\#,\natural}(\mathcal{E})$, both $I(X)$ and $P(X)$ lie in $C_N^{\#,\natural}(\mathcal{E})$ due to Construction 1.51 and Remark 2.10.(b), and in $\operatorname{Proj}(C_N(\mathcal{E}))$ due to Lemma 1.50. Since also ΣX and $\Sigma^{-1}X$ lie in $C_N^{\#,\natural}(\mathcal{E})$ due to Construction 1.58.(a) and Corollary 2.18, the morphisms i_X and p_X from Construction 1.51 are admissible in $C_N^{\#,\natural}(\mathcal{E})$, and part (b) follows. In view of Theorem 1.54, the assumptions of Lemma 1.36.(b) are then satisfied, and part (c) follows. Intersecting with $C_N(\mathcal{A})$, the preceding arguments also prove the more general claims. □

Similar arguments yield

Proposition 2.21. *Let $\mathcal{E}'$ be an exact subcategory of $\mathcal{E}$. Consider subcategories $\mathcal{A}$ and $\mathcal{A}'$ of $\mathcal{E}$ and $\mathcal{E}'$, respectively, and suppose that $\mathcal{A}'$ is a subcategory of $\mathcal{A}$. Then $C_N^{\#,\natural_{\mathcal{E}'}}(\mathcal{A}')$ is a fully exact sub-Frobenius category of $C_N^{\#,\natural_{\mathcal{E}}}(\mathcal{A})$, and there is a canonical fully faithful, triangulated functor $\mathcal{K}_N^{\#,\natural_{\mathcal{E}'}}(\mathcal{A}') \longrightarrow \mathcal{K}_N^{\#,\natural_{\mathcal{E}}}(\mathcal{A})$ where $\# \in \{\infty, +, -, b\}$ and $\natural \in \{\infty, +, -, b, \varnothing, \varnothing^*\}$.* □

Corollary 2.22. *Let $\mathcal{E}'$ be an exact subcategory of $\mathcal{E}$ with* $\mathrm{Proj}(\mathcal{E}') \subseteq \mathrm{Proj}(\mathcal{E})$. *Then* $\mathrm{APC}_N(\mathcal{E}')$ *is a fully exact sub-Frobenius category of* $\mathrm{APC}_N(\mathcal{E})$ *and there is a canonical fully faithful, triangulated functor* $\underline{\mathrm{APC}}_N(\mathcal{E}') \to \underline{\mathrm{APC}}_N(\mathcal{E})$. *The verbatim statement holds for* TAPC *instead of* APC. ☐

2.3. **Acyclic N-arrays.** The kernels in an acyclic N-complex can be organized in an array of bicartesian squares. Over an Abelian category, such arrays have been used for instance in [BM24] and [IKM17]. We build *acyclic N-arrays* from acyclic N-complexes over an exact category and establish an equivalence between the respective categories (Theorem 2.26). Our construction is local with respect to the indices of the considered N-complex (Proposition 2.30). As a result, one can define *soft truncations* as in the Abelian case (Definition 2.31).

Definition 2.23. A **(bicartesian) N-array** $X_\bullet^\bullet = (X_\bullet^\bullet, p_\bullet^\bullet, i_\bullet^\bullet)$ over an exact category, where $X_\bullet^\bullet = (X_r^k)_{r\in\{0,\dots,N\}}^{k\in\mathbb{Z}}$, $p_\bullet^\bullet = (p_r^k)_{r\in\{1,\dots,N\}}^{k\in\mathbb{Z}}$ and $i_\bullet^\bullet = (i_r^k)_{r\in\{0,\dots,N-1\}}^{k\in\mathbb{Z}}$, is a diagram consisting of (bicartesian) commutative squares

$$\begin{tikzcd}
& X_{r+1}^{k+1} \arrow[dr, "p_{r+1}^{k+1}"] & \\
X_r^k \arrow[ur, "i_r^k"] \arrow[dr, "p_r^k"'] & & X_r^{k+1} \\
& X_{r-1}^k \arrow[ur, "i_{r-1}^k"'] &
\end{tikzcd}$$

for $k \in \mathbb{Z}$ and $r \in \{1, \dots, N-1\}$. We call $X_\bullet^\bullet$ **epic**, resp. **monic**, if p_r^k is an admissible epic, resp. i_r^k an admissible monic, for all k and r. We call it **bounded above**, **bounded below**, or **bounded**, if $X_\bullet^k = 0$ for any sufficiently large, any sufficiently small, almost all $k \in \mathbb{Z}$, respectively.
A morphism $f\colon X \to Y$ between such arrays is a collection $f_\bullet^\bullet = (f_r^k)_{r\in\{0,\dots,N\}}^{k\in\mathbb{Z}}$ of morphisms $f_r^k\colon X_r^k \to Y_r^k$ which establish commutativity. We drop the bullets and write $X = (X, p, i)$ if there is no ambiguity.

Remark 2.24. Any N-array X over an exact category $\mathcal{E}$ gives rise to morphisms $p^{\{N\}}\colon X_N^\bullet \to X_0^\bullet$ and $i^{\{N\}}\colon X_0^\bullet \to X_N^\bullet$ in $C(\mathcal{E})$, where $d_{X_0^\bullet} = pi$ and $d_{X_N^\bullet} = ip$. Note that $X_0^\bullet, X_N^\bullet \in C_N(\mathcal{E})$ if $p^{\{N\}} = 0$ or $i^{\{N\}} = 0$.

Definition 2.25. An **acyclic N-array** (of $X_N^\bullet \in C_N(\mathcal{E})$) over an exact category $\mathcal{E}$ is an epic and monic bicartesian N-array with $X_0^\bullet = 0$, see also [BM24, Def. 4.4]. We denote the category of acyclic N-arrays over $\mathcal{E}$ by $A_N(\mathcal{E})$.

Over an Abelian category, Theorem 2.26 can be easily verified as mentioned in [BM24, p. 21].

Theorem 2.26. *For an exact category $\mathcal{E}$, the categories $A_N(\mathcal{E})$ and $C_N^{\infty,\varnothing}(\mathcal{E})$ are equivalent. Under this equivalence, an acyclic N-complex $X \in C_N^{\infty,\varnothing}(\mathcal{E})$ corresponds to an acyclic N-array $(X_r^k)_{r=0,\dots,N}^{k\in\mathbb{Z}}$ of X, where $X_r^k = C_{(r)}^k(X) = Z_{(r)}^{k+N-r}(X)$ and, in particular, $X_N^\bullet = X$.*

Notation 2.27.

(a) By $A \hookrightarrow B$ (in contrast to $A \rightarrowtail B$), we denote a monic (which might not be admissible).

(b) By $A \twoheadrightarrow B$ (in contrast to $A \twoheadrightarrow B$), we denote an epic (which might not be admissible).

Lemma 2.28. *Let $\mathcal{E}$ be an exact category.*

(a) Suppose that the following diagram (XY) in $\mathcal{E}$ is bicartesian:

$$\begin{tikzcd}
 & & \bullet \arrow[dr] & \\
 & \bullet \arrow[ur, tail, "i"] \arrow[dr, two heads, "e"] & (Y) & \bullet \\
\bullet \arrow[ur] \arrow[dr] & (X) & \bullet \arrow[ur] & \\
 & \bullet \arrow[ur] & &
\end{tikzcd}$$

Then (Y) is bicartesian. If e is an admissible epic, then also (X) is bicartesian.

(b) Suppose that the following diagram $(Y'X')$ in $\mathcal{E}$ is bicartesian:

$$\begin{tikzcd}
 & \bullet \arrow[dr, two heads, "p"] & & \\
\bullet \arrow[ur] \arrow[dr] & (Y') & \bullet \arrow[dr] & \\
 & \bullet \arrow[ur, hook, "m"] \arrow[dr] & (X') & \bullet \\
 & & \bullet \arrow[ur] &
\end{tikzcd}$$

Then (Y') is bicartesian. If m is an admissible monic, then also (X') is bicartesian.

Proof.

(a) As e is an epic, (Y) is a pushout by Lemma 1.3.(a) and hence bicartesian by Corollary 1.13.(a). As i is a monic, (X) is a pullback by Lemma 1.3.(b) and hence bicartesian by Corollary 1.13.(a) if e is an admissible epic.

(b) is dual to (a). □

Lemma 2.29. *Over an exact category $\mathcal{E}$, consider the commutative diagram*

$$\begin{tikzcd}[column sep=tiny]
X_N^0 \arrow[dr, ->>] & & X_N^1 \arrow[dr, ->>] & & X_N^2 \arrow[dr, ->>] & & X_N^3 & \cdots & X_N^{N-3} \arrow[dr, ->>] & & X_N^{N-2} \arrow[dr, ->>] & & X_N^{N-1} \arrow[dr, ->>] & & X_N^N \\
& X_{N-1}^0 \arrow[ur, tail] \arrow[dr, ->>] & & X_{N-1}^1 \arrow[ur, tail] \arrow[dr, ->>] & & X_{N-1}^2 \arrow[ur, tail] \arrow[dr, ->>] & & \cdots & & X_{N-1}^{N-3} \arrow[ur, tail] \arrow[dr, ->>] & & X_{N-1}^{N-2} \arrow[ur, tail] \arrow[dr, ->>] & & X_{N-1}^{N-1} \arrow[ur, tail] & \\
& & X_{N-2}^0 \arrow[ur, tail] \arrow[dr, ->>] & & X_{N-2}^1 \arrow[ur, tail] \arrow[dr, ->>] & & \ddots \arrow[dr, ->>] & & \iddots \arrow[ur, tail] & & X_{N-2}^{N-3} \arrow[ur, tail] \arrow[dr, ->>] & & X_{N-2}^{N-2} \arrow[ur, tail] & & \\
& & & X_{N-3}^0 \arrow[ur, tail] \arrow[dr, ->>] & & \ddots \arrow[dr, ->>] & & X_4^2 \arrow[ur, tail] \arrow[dr, ->>] & & \iddots \arrow[ur, tail] & & X_{N-3}^{N-3} \arrow[ur, tail] & & & \\
& & & & \ddots \arrow[dr, ->>] & & X_3^1 \arrow[ur, tail] \arrow[dr, ->>] & & X_3^2 \arrow[ur, tail] \arrow[dr, ->>] & & \iddots \arrow[ur, tail] & & & & \\
& & & & & X_2^0 \arrow[ur, tail] \arrow[dr, ->>] & & X_2^1 \arrow[ur, tail] \arrow[dr, ->>] & & X_2^2 \arrow[ur, tail] & & & & & \\
& & & & & & X_1^0 \arrow[ur, tail] \arrow[dr, ->>] & & X_1^1 \arrow[ur, tail] & & & & & & \\
& & & & & & & X_0^0. \arrow[ur, tail] & & & & & & &
\end{tikzcd}$$

For $n \in \{1, \ldots, N-1\}$ *and* $r \in \{n, \ldots, N-1\}$, *let* (Y_r^n) *denote the square*

$$\begin{tikzcd}
& X_{r+1}^n \arrow[dr, ->>] & \\
X_r^{n-1} \arrow[ur, tail] \arrow[dr, ->>] & (Y_r^n) & X_r^n \\
& X_{r-1}^{n-1}. \arrow[ur, tail] &
\end{tikzcd}$$

Suppose that for each $n \in \{1, \ldots, N-1\}$ *the concatenation of*

$$\begin{tikzcd}
& & (Y_{N-1}^n) & & \\
& \iddots & & \ddots & \\
(Y_{N-n}^1) & & & & (Y_n^n) \\
& \ddots & & \iddots & \\
& & (Y_1^1) & &
\end{tikzcd}$$

is bicartesian. Then each individual square (Y_r^n) *is bicartesian with admissible morphisms.*

Proof. By Lemma 2.28.(a), the assumption implies that $(Y_{N-1}^n \cdots Y_n^n)$ bicartesian. We prove the claim by descending induction on $r = N-1, \ldots, n$. Starting with $r = N-1$, the square (Y_{N-1}^n) is bicartesian due to Lemma 2.28.(b), and its morphisms are admissible by Corollary 1.13.(b). Let now $n, n' \in \{1, \ldots, N-1\}$ and $r, r' \in \{n, \ldots, N-2\}$, and suppose that the claim holds for $(Y_{r'}^{n'})$ whenever $r' > r$. Then $(Y_{N-1}^n \cdots Y_{r+1}^n)$ is bicartesian with admissible morphisms, see Remark 1.2, and the upper morphisms of (Y^n_r) are admissible. So, $(Y^n_r \cdots Y^n_n)$ is bicartesian by the additional claim of Lemma 2.28.(b). Now the claim for (Y^n_r) follows from Lemma 2.28.(b) and Corollary 1.13.(b) as before. □

Proposition 2.30. *Let $X \in C_N(\mathcal{E})$ be an N-complex over an exact category $\mathcal{E}$ and $n \in \mathbb{Z}$. Suppose that all compositions of differentials between positions in $\{n, n+1, \dots, n+N\}$ are admissible.*

(a) For each choice X^k_r of (co)images, see Remark 1.39.(b), there is a unique commutative diagram:

$$\begin{tikzcd}
X^n \arrow[rr] \arrow[dr, two heads] & & X^{n+1} \arrow[rr] \arrow[dr, two heads] & & X^{n+2} \arrow[r] & \cdots \arrow[r] & X^{n+N-2} \arrow[rr] \arrow[dr, two heads] & & X^{n+N-1} \arrow[rr] \arrow[dr, two heads] & & X^{n+N} \\
& X^n_{N-1} \arrow[ur, tail] \arrow[dr, two heads] & & X^{n+1}_{N-1} \arrow[ur, tail] \arrow[dr, two heads] & & & & X^{n+N-2}_{N-1} \arrow[ur, tail] \arrow[dr, two heads] & & X^{n+N-1}_{N-1} \arrow[ur, tail] & \\
& & X^n_{N-2} \arrow[ur, hook] \arrow[dr, two heads] & & \ddots \arrow[dr, two heads] & & \iddots \arrow[ur, hook] & & X^{n+N-2}_{N-2} \arrow[ur, hook] & & \\
& & & \ddots \arrow[dr, two heads] & & X^{n+1}_2 \arrow[ur, hook] \arrow[dr, two heads] & & \iddots \arrow[ur, hook] & & & \\
& & & & X^n_1 \arrow[ur, hook] \arrow[dr, two heads] & & X^{n+1}_1 \arrow[ur, hook] & & & & \\
& & & & & 0 \arrow[ur, hook] & & & & &
\end{tikzcd}$$

If X acyclic at position $k \in \{n, \dots, n+N\}$, we have $X^k_r = C^k_{(r)}(X)$ and $X^{k-N+r}_r = Z^k_{(r)}(X)$, for all occurring r. If this holds for all k, then every morphism of such N-complexes induces a unique morphism of the associated diagrams.

(b) If $\gamma^{n+k}_{N-k}(X)$ is 2-acyclic at position $n+k$ for each $k \in \{1, \dots, N-1\}$, then all squares of the diagram in in (a) are bicartesian with admissible morphisms. In particular, this holds if X is acyclic at positions $n+1, \dots, n+N-1$.

Proof.

(a) The morphisms in the diagram exist due to the universal property of (co)images. It commutes by repeated application of Remark 2.7. The morphisms $f^k_r \colon X^k_r \to Y^k_r$ induced by f on cokernels form a morphism of the respective diagrams associated to X and Y. Indeed, using Remark 2.7 multiple times, one verifies that the following diagrams commute for all occurring indices k, r and s:

$$\begin{tikzcd}
X^k \arrow[dd, "f^k"] \arrow[dr, two heads] & & \\
 & X^k_r \arrow[dd, "f^k_r"] \arrow[dr, two heads] & \\
Y^k \arrow[dr, two heads] & & X^k_s \arrow[dd, "f^k_s"] \\
 & Y^k_r \arrow[dr, two heads] & \\
 & & Y^k_s
\end{tikzcd}
\qquad
\begin{tikzcd}
X^k \arrow[rr, "d_X^{\{N-r\}}", "\circ" marking] \arrow[dr, two heads] \arrow[ddd, "f^k"] & & X^{k+N-r} \arrow[ddd, "f^{k+N-r}"] \\
 & X^k_r \arrow[ur, tail] \arrow[d, "f^k_r"] & \\
 & Y^k_r \arrow[dr, tail] & \\
Y^k \arrow[ur, two heads] \arrow[rr, "d_Y^{\{N-r\}}", "\circ" marking] & & Y^{k+N-r}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & X^k \arrow[dd, "f^k"] \\
 & X^{k-N+r}_r \arrow[ur, tail] \arrow[dd, "f^{k-N+r}_r"] & \\
X^{k-N+s}_s \arrow[ur, tail] \arrow[dd, "f^{k-N+s}_s"] & & Y^k \\
 & Y^{k-N+r}_r \arrow[ur, tail] & \\
Y^{k-N+s}_s \arrow[ur, tail] & &
\end{tikzcd}$$

(b) By hypothesis, $X^n_{N-k} \rightarrowtail X^{n+k} \twoheadrightarrow X^{n+k}_k$ is short exact, and hence

$$\begin{tikzcd}
 & X^{n+k} \arrow[dr, two heads] & \\
X^n_{N-k} \arrow[ur, tail] \arrow[dr, two heads] & & X^{n+k}_k \\
 & 0 \arrow[ur, tail] &
\end{tikzcd}$$

is bicartesian for all $k \in \{1, \dots, N-1\}$ by Proposition 1.12. Lemma 2.29 yields the claim. □

Proof of Theorem 2.26. Given an acyclic N-array X, the sequences $X^{n-r}_{N-r} \rightarrowtail X^n_N \twoheadrightarrow X^n_r$ are exact for all n and r by Proposition 1.12. So, $X^\bullet_N$ is an acyclic N-complex, see Remark 2.24, with $C^n_{(r)}(X^\bullet_N) = Z^{n+N-r}_{(r)}(X^\bullet_N) = X^n_r$ for all n and r. This yields a functor $A_N(\mathcal{E}) \to C^{\infty,\varnothing}_N(\mathcal{E})$ which sends a morphism $f\colon X \to Y$ in $A_N(\mathcal{E})$ to the morphism $f^\bullet_N\colon X^\bullet_N \to Y^\bullet_N$ in $C_N(\mathcal{E})$.
To show essential surjectivity, let $X \in C^{\infty,\varnothing}_N(\mathcal{E})$ be an acyclic N-complex. Apply Proposition 2.30 for all $n \in \mathbb{Z}$ with a fixed choice of all (co)images. The respective diagrams patch together to form an acyclic N-array, sent to X under the above functor. Fullness follows from Proposition 2.30 as well. To see that the functor is faithful, suppose that $f^n_N = 0$ for all $n \in \mathbb{Z}$. Then the commutative diagram

$$\begin{tikzcd}
X^n_N \arrow[r, two heads] \arrow[d, "f^n_N"] \arrow[dr, "0"] & X^n_r \arrow[d, "f^n_r"] \\
Y^n_N \arrow[r, two heads] & Y^n_r
\end{tikzcd}$$

shows that $f^n_r = 0$ for all n and r. □

Based on Proposition 2.30, we define soft truncations. For convenience, we impose slightly stronger acyclicity hypotheses than necessary.

Definition 2.31. Let $X \in C_N(\mathcal{E})$ be an N-complex over an exact category $\mathcal{E}$ and $n \in \mathbb{Z}$.

(a) If X is acyclic at positions $n, \dots, n+N-2$, its **(left) soft truncation** is the N-complex

$$\sigma^{\geq n}X\colon \qquad C^n_{(1)} \rightarrowtail \cdots \rightarrowtail C^{n+N-2}_{(N-1)} \rightarrowtail X^{n+N-1} \longrightarrow X^{n+N} \longrightarrow X^{n+N+1} \longrightarrow \cdots .$$

Note that $C^{n+r-1}_{(r)}(X) = Z^{n+N-1}_{(r)}(X)$ for all $r \in \{1, \dots, N-1\}$ if X is acyclic also at position $n + N - 1$.

(b) If X is acyclic at positions $n - N + 2, \dots, n$, its **(right) soft truncation** is the N-complex

$$\sigma^{\leq n}X\colon \qquad \cdots \longrightarrow X^{n-N-1} \longrightarrow X^{n-N} \longrightarrow X^{n-N+1} \twoheadrightarrow Z^{n-N+2}_{(N-1)} \twoheadrightarrow \cdots \twoheadrightarrow Z^{n}_{(1)}.$$

Note that $Z^{n-r+1}(X) = C^{n-N+1}_{(r)}(X)$ for all $r \in \{1, \dots, N-1\}$ if X is acyclic also at position $n - N + 1$.

Corollary 2.32. *Let $X \in C_N(\mathcal{E})$ be an N-complex over an exact category $\mathcal{E}$ and $n \in \mathbb{Z}$.*

(a) If X is acyclic at all positions greater than or equal to n, then $\sigma^{\geq n}X$ is acyclic. Dually, if X is acyclic at all positions up to n, then $\sigma^{\leq n}X$ is acyclic.

*(b) If X is acyclic at positions $n - N + 1, \dots, n + 1 \in \mathbb{Z}$, then there is a term*wise *short exact sequence of N-complexes*

$$\begin{tikzcd}
\sigma^{\leq n}X\colon \arrow[d] & \cdots \arrow[r] & X^{n-N+1} \arrow[r, two heads] \arrow[d, tail, "\mathrm{id}"] & X^{n-N+1}_{N-1} \arrow[r, two heads] \arrow[d, tail] & \cdots \arrow[r, two heads] & X^{n-N+1}_{1} \arrow[r] \arrow[d, tail] & 0 \arrow[r] \arrow[d, tail] & \cdots \\
X\colon \arrow[d] & \cdots \arrow[r] & X^{n-N+1} \arrow[r] \arrow[d, two heads] & X^{n-N+2} \arrow[r] \arrow[d, two heads] & \cdots \arrow[r] & X^{n} \arrow[r] \arrow[d, two heads] & X^{n+1} \arrow[r] \arrow[d, two heads, "\mathrm{id}"] & \cdots \\
\sigma^{\geq n-N+2}X\colon & \cdots \arrow[r] & 0 \arrow[r] & X^{n-N+2}_{1} \arrow[r, tail] & \cdots \arrow[r, tail] & X^{n}_{N-1} \arrow[r, tail] & X^{n+1} \arrow[r] & \cdots ,
\end{tikzcd}$$

where $X^k_r = C^k_{(r)}(X) = Z^{k+N-r}_{(r)}(X)$ for all occurring indices k and r.

Proof. Part (b) is obvious from the definitions. To see (a), patch all diagrams that can be obtained from Proposition 2.30, as in the proof of Theorem 2.26. Then modify the resulting diagram at the left end as follows, and extend it by zero to create an acyclic N-array of $\sigma^{\geq n}X$:

$$\begin{tikzcd}[column sep=tiny]
X^n_1 \arrow[rr, tail] \arrow[dr, equal] & & X^{n+1}_2 \arrow[rr, tail] \arrow[dr, equal] & & X^{n+2}_3 \arrow[r, tail, dash] & \cdots \arrow[r] & X^{n+N-3}_{N-2} \arrow[rr, tail] \arrow[dr, equal] & & X^{n+N-2}_{N-1} \arrow[rr, tail] \arrow[dr, equal] & & X^{n+N-1} \arrow[rr] \arrow[dr, two heads] & & X^{n+N} \arrow[r] \arrow[dr, two heads] & \cdots \\
& X^n_1 \arrow[ur, tail] \arrow[dr, equal] & & X^{n+1}_2 \arrow[ur, tail] \arrow[dr, equal] & & & & X^{n+N-3}_{N-2} \arrow[ur, tail] \arrow[dr, equal] & & X^{n+N-2}_{N-1} \arrow[ur, tail] \arrow[dr, two heads] & & X^{n+N-1}_{N-1} \arrow[ur, tail] \arrow[dr, two heads] & & \ddots \\
& & X^n_1 \arrow[ur, tail] \arrow[dr, equal] & & \ddots \arrow[dr, equal] & & \iddots \arrow[ur, tail] & & X^{n+N-3}_{N-2} \arrow[ur, tail] \arrow[dr, two heads] & & X^{n+N-2}_{N-2} \arrow[ur, tail] \arrow[dr, two heads] & & \ddots & \\
& & & \ddots \arrow[dr, equal] & & X^{n+1}_2 \arrow[ur, tail] \arrow[dr, equal] & & \iddots \arrow[ur, tail] & & X^{n+N-3}_{N-3} \arrow[ur, tail] \arrow[dr, two heads] & & \ddots & & \\
& & & & X^n_1 \arrow[ur, tail] \arrow[dr, equal] & & X^{n+1}_2 \arrow[ur, tail] \arrow[dr, two heads] & & \iddots \arrow[ur, tail] & & \ddots & & & \\
& & & & & X^n_1 \arrow[ur, tail] \arrow[dr, two heads] & & X^{n+1}_1 \arrow[ur, tail] \arrow[dr, two heads] & & \iddots & & & & \\
& & & & & & 0 \arrow[ur, tail] & & 0 \arrow[ur, tail] & & & & &
\end{tikzcd}$$

The claimed acyclicity follows from Theorem 2.26. □

2.4. **Resolutions of N-complexes.** Keller described (injective) resolutions of 2-complexes over an exact category $\mathcal{E}$ which are bounded on one side, see [Kel90, 4.1, Lemma]. In this subsection we generalize his approach to construct projectively resolving N-arrays, which then yield projective N-resolutions (Corollary 2.43). For elements of $\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E})$, see Notation 1.69, such resolutions take the form of one-sided acyclic N-arrays (Corollary 2.38).

Definition 2.33. Let $\mathcal{E}$ be an exact category.

(a) We refer to an epic bicartesian N-array (X, p, i) over $\mathcal{E}$ with $i^{\{N\}} = 0$ as a **resolving N-array** (of $X^\bullet_0 \in C_N(\mathcal{E})$). We call it **projectively resolving** if $X^\bullet_N \in C_N(\mathrm{Proj}(\mathcal{E}))$, see Remark 2.24.

(b) We refer to a monic bicartesian N-array (X, p, i) over $\mathcal{E}$ with $p^{\{N\}} = 0$ as a **coresolving N-array** (of $X^\bullet_0 \in C_N(\mathcal{E})$). We call it **injectively coresolving** if $X^\bullet_0 \in C_N(\mathrm{Inj}(\mathcal{E}))$, see Remark 2.24.

In the following, we consider only (projectively) resolving N-arrays. However, there are obvious dual statements on (injectively) coresolving N-arrays.

Proposition 2.34. *Let $\mathcal{E}$ be an exact category with enough projectives. Then any bounded above N-complex over $\mathcal{E}$ admits a bounded above, projectively resolving N-array.*

The proof relies on the following argument of Keller:

Lemma 2.35. *Consider a commutative diagram*

$$\begin{tikzcd}
 & & \bullet \arrow[dr, two heads, "b"] & & \bullet \arrow[dr, two heads, "d"] & \\
 & \bullet \arrow[ur, "a'"] \arrow[dr, two heads, "b'"] & \diamond & \bullet \arrow[ur, "c'"] \arrow[dr, two heads, "d'"] & \diamond & \bullet \\
\bullet \arrow[ur, dashed, "j"] \arrow[rr, "e'"] & & \bullet \arrow[ur, "a"] \arrow[rr, "e"] & & \bullet \arrow[ur, "c"] &
\end{tikzcd}$$

of two bicartesian squares in an additive category. If $c'a = 0$ and $ee' = 0$, then there exists a morphism j completing the diagram with $a'j = 0$.

Proof. Interpreting the bicartesian squares as short exact sequences as in Proposition 1.12.(b), the hypotheses yield

$$\begin{pmatrix} c' \\ d' \end{pmatrix} \begin{pmatrix} -b & a \end{pmatrix} \begin{pmatrix} 0 \\ e' \end{pmatrix} = \begin{pmatrix} c'ae' \\ d'ae' \end{pmatrix} = \begin{pmatrix} c'ae' \\ ee' \end{pmatrix} = 0,$$

and hence $\begin{pmatrix} -b & a \end{pmatrix} \begin{pmatrix} 0 \\ e' \end{pmatrix} = 0$, since $\begin{pmatrix} c' \\ d' \end{pmatrix}$ is a monic. Then j exists by the universal property of pullbacks. □

Proof of Proposition 2.34. Let $X \in C^-_N(\mathcal{E})$, and set $m := \max\{k \in \mathbb{Z} \mid X^k \neq 0\}$. We construct a projectively resolving N-array $X^\bullet_\bullet$ of X as follows:

$$\begin{tikzcd}[column sep=tiny]
X_N^k \arrow[rr] \arrow[dr, two heads] & & X_N^{k+1} \arrow[rr] \arrow[dr, two heads] & & X_N^{k+2} \arrow[rr] & & \cdots \arrow[rr] & & X_N^{m-3} \arrow[rr] & & X_N^{m-2} \arrow[rr] \arrow[dr, two heads] & & X_N^{m-1} \arrow[rr] \arrow[dr, two heads] & & X_N^m \arrow[dr, two heads] \\
& X_{N-1}^k \arrow[ur] \arrow[dr, two heads] & (Y_{N-1}^{k+1}) & X_{N-1}^{k+1} \arrow[ur] \arrow[dr, two heads] & & & & & & \cdots \arrow[ur] & & X_{N-1}^{m-2} \arrow[ur] \arrow[dr, two heads] & (Y_{N-1}^{m-1}) & X_{N-1}^{m-1} \arrow[ur] \arrow[dr, two heads] & (Y_{N-1}^m) & X^m \arrow[dr, equal] \\
& & X_{N-2}^k \arrow[ur] \arrow[dr, two heads] & & \cdots \arrow[dr, two heads] & & & & & & \cdots \arrow[ur] & & X_{N-2}^{m-2} \arrow[ur] \arrow[dr, two heads] & (Y_{N-2}^{m-1}) & X^{m-1} \arrow[ur] \arrow[dr, equal] & & \cdots \arrow[dr, equal] \\
& & & \cdots \arrow[dr, two heads] & & X_3^{k+1} \arrow[dr, two heads] & & & & & & \cdots \arrow[ur] & & X^{m-2} \arrow[ur] \arrow[dr, equal] & & \cdots \arrow[dr, equal] & & X^m \arrow[dr, equal] \\
& & \cdots \arrow[dr, two heads] & & X_2^k \arrow[ur] \arrow[dr, two heads] & (Y_2^{k+1}) & X_2^{k+1} \arrow[dr, two heads] & & & & \cdots \arrow[dr, two heads] & & \cdots \arrow[ur] & & \cdots \arrow[dr, equal] & & X^{m-1} \arrow[ur] \arrow[dr, equal] & (Y_2^m) & X^m \arrow[dr, equal] \\
& & & X_1^{k-1} \arrow[ur] \arrow[dr, two heads] & (Y_1^k) & X_1^k \arrow[ur] \arrow[dr, two heads] & (Y_1^{k+1}) & X_1^{k+1} \arrow[dr, two heads] & & & & X^{m-N+2} \arrow[ur] & & \cdots \arrow[dr, equal] & & X^{m-2} \arrow[ur] \arrow[dr, equal] & (Y_1^{m-1}) & X^{m-1} \arrow[ur] \arrow[dr, equal] & (Y_1^m) & X^m \arrow[dr, equal] \\
\cdots \arrow[rr] & & X^{k-2} \arrow[ur] \arrow[rr] & & X^{k-1} \arrow[ur] \arrow[rr] & & X^k \arrow[ur] \arrow[rr] & & X^{k+1} \arrow[r, dash] & \cdots \arrow[r] & X^{m-N+1} \arrow[ur] \arrow[rr] & & \cdots \arrow[rr] & & X^{m-3} \arrow[ur] \arrow[rr] & & X^{m-2} \arrow[ur] \arrow[rr] & & X^{m-1} \arrow[ur] \arrow[rr] & & X^m
\end{tikzcd}$$

Figure 1. Keller's resolution for general N

For all $k > m$ and $r \in \{1, \dots, N-1\}$, set $X_r^k := 0$. Fix $n \in \mathbb{Z}_{\leq m}$ and assume that (Y_r^k) has already been constructed for $r \in \{1, \dots, N\}$ and $k > n$. Consider the following diagram joining the concatenated bicartesian squares $(Y_1^{n+1} \cdots Y_{N-1}^{n+N-1})$ and $(Y_{N-1}^{n+N} \cdots Y_1^{n+N})$, see Remark 1.2:

$$\begin{tikzcd}
 & & & X_N^{n+N-1} \arrow[dr, "p_N^{n+N-1}"] & & X_N^{n+N} \arrow[ddrr, two heads, "\prod_{l=2}^{N} p_l^{n+N}"] & & \\
 & & & & X_{N-1}^{n+N-1} \arrow[ur, "i_{N-1}^{n+N-1}"'] \arrow[ddrr, two heads, "\prod_{l=1}^{N-1} p_l^{n+N-1}"] & & & \\
 & X_1^n \arrow[uurr, "\prod_{l=N-1}^{1} i_l^{n-1+l}"] \arrow[dr, two heads, "p_1^n"] & & & & & & X_1^{n+N} \\
X^{n-1} \arrow[ur, dashed, "i_0^{n-1}"] \arrow[rr, "d^{n-1}"] & & X^n \arrow[uurr, "\prod_{l=N-2}^{0} i_l^{n+l}"] \arrow[rrrr, "d^{\{N-1\}}"] & & & & X^{n+N-1} \arrow[ur, "i_0^{n+N-1}"'] &
\end{tikzcd}$$

Then the morphism i_0^{n-1} exists due to Lemma 2.35. The squares (Y_r^n) can now be defined by successive pullbacks for increasing $r = 1, \dots, N-1$. To complete the induction step, we pick an admissible epic $p_N^{n-1} \colon X_N^{n-1} \twoheadrightarrow X_{N-1}^{n-1}$ with $X_N^{n-1} \in \mathrm{Proj}(\mathcal{E})$. □

Remark 2.36. Let X be a (projectively) resolving N-array over an exact category $\mathcal{E}$. Pick $n \in \mathbb{Z}$ and $r \in \{1, \dots, N-1\}$. Then suitably composing morphisms in X yields a (projectively) resolving 2-array of $\gamma_r^n(X_0^\bullet)$, as constructed by Keller:

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[rr] & & X_N^{n-N+r} \arrow[rr, "d_{X_N}^{\{N-r\}}"] \arrow[dr, two heads] & & X_N^n \arrow[rr, "d_{X_N}^{\{r\}}"] \arrow[dr, two heads] & & X_N^{n+r} \arrow[rr, "d_{X_N}^{\{N-r\}}"] \arrow[dr, two heads] & & X_N^{n+N} \arrow[rr] & & \cdots \\
 & \cdots & & X_r^{n-N+r} \arrow[ur] \arrow[dr, two heads] & & X_{N-r}^n \arrow[ur] \arrow[dr, two heads] & & X_r^{n+r} \arrow[ur] \arrow[dr, two heads] & & \cdots & \\
\cdots \arrow[rr] & & X_0^{n-N} \arrow[rr, "d_{X_0^\bullet}^{\{r\}}"] \arrow[ur] & & X_0^{n-N+r} \arrow[rr, "d_{X_0^\bullet}^{\{N-r\}}"] \arrow[ur] & & X_0^n \arrow[rr, "d_{X_0^\bullet}^{\{r\}}"] \arrow[ur] & & X_0^{n+r} \arrow[rr] & & \cdots
\end{tikzcd}$$

Remark 2.37. Let $X_\bullet^\bullet$ and $Y_\bullet^\bullet$ be resolving N-arrays of N-complexes X and Y, respectively, over an exact category $\mathcal{E}$. Suppose $X_\bullet^\bullet$ is projectively resolving and one of $X_\bullet^\bullet$ and $Y_\bullet^\bullet$ is bounded above. Then any morphism $X \to Y$ in $C_N(\mathcal{E})$ lifts to a morphism $X_\bullet^\bullet \to Y_\bullet^\bullet$, which in turn induces a morphism $X_N^\bullet \to Y_N^\bullet$ in $C_N(\mathcal{E})$: Due to boundedness, the morphism $X_r^k \to Y_r^k$ must be zero, for any sufficiently large $k \in \mathbb{Z}$. The remaining morphisms exist exist due to functoriality of pullbacks for $r \in \{1, \dots, N-1\}$, and due to projectivity for $r = N$.

Combining the proof of Proposition 2.34 with Propositions 1.12 and 1.14 yields

Corollary 2.38. *Let $\mathcal{E}$ be an exact category with enough projectives and $X \in \mathrm{Mor}_{N-2}^{\mathrm{m}}(\mathcal{E})$. Then $\iota^0 X$, see Notation 1.69, admits a projectively resolving N-array which can be modified into a one-sided N-acyclic array as follows:*

$$\begin{tikzcd}
\cdots \arrow[r] & X_N^{-N+1} \arrow[rr] \arrow[dr, two heads] & & X_N^{-N+2} \arrow[rr] \arrow[dr, two heads] & & X_N^{-N+3} \arrow[r] & \cdots \arrow[r] & X_N^{-1} \arrow[rr] \arrow[dr, two heads] & & X_N^{0} \arrow[dr, two heads] & \\
& & X_{N-1}^{-N+1} \arrow[ur, tail] \arrow[dr, two heads] & & X_{N-1}^{-N+2} \arrow[ur, tail] \arrow[dr, two heads] & & & & X_{N-1}^{-1} \arrow[ur, tail] \arrow[dr, two heads] & & X^{N-1} \\
& \iddots \arrow[ur, tail] & & X_{N-2}^{-N+1} \arrow[ur, tail] \arrow[dr, two heads] & & \ddots \arrow[dr, two heads] & & \iddots \arrow[ur, tail] & & X^{N-2} \arrow[ur, tail] & \\
& & \iddots \arrow[ur, tail] & & \ddots \arrow[dr, two heads] & & X_2^{-N+2} \arrow[ur, tail] \arrow[dr, two heads] & & \iddots \arrow[ur, tail] & & \\
& & & & & X_1^{-N+1} \arrow[ur, tail] \arrow[dr, two heads] & & X^{1} \arrow[ur, tail] \arrow[dr, two heads] & & & \\
& & & & 0 \arrow[ur, tail] & & 0 \arrow[ur, tail] & & 0 & &
\end{tikzcd}$$

In particular, X can be recovered as a cokernel in $\operatorname{Mor}_{N-2}(\mathcal{E})$ *of the morphism*

$$\begin{tikzcd}
X_N^{-N+1} \arrow[r, equal] \arrow[d] & X_N^{-N+1} \arrow[r, equal] \arrow[d] & \cdots \arrow[r, equal] & X_N^{-N+1} \arrow[r, equal] \arrow[d] & X_N^{-N+1} \arrow[d] \\
X_N^{-N+2} \arrow[r] & X_N^{-N+3} \arrow[r] & \cdots \arrow[r] & X_N^{-1} \arrow[r] & X_N^{0}.
\end{tikzcd}$$

The obvious dual statements hold for elements of $\operatorname{Mor}^{\mathrm{e}}_{N-2}(\mathcal{E})$. □

Remark 2.39. In the situation of Corollary 2.38, the induced morphism $X_N^\bullet \to Y_N^\bullet$ in Remark 2.37 can be considered as a lift of a morphism $X \to Y$ in $\operatorname{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E})$.

Proposition 2.40. *Let $\mathcal{E}$ be an exact category with enough projectives. For each $X \in \operatorname{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E})$, choose an N-complex $X_N^\bullet \in C_N^-(\operatorname{Proj}(\mathcal{E}))$ as in Corollary 2.38. Then morphisms lift uniquely up to homotopy from $\operatorname{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E})$ to $C_N^-(\operatorname{Proj}(\mathcal{E}))$, see Remark 2.39. This defines a fully faithful functor $\rho : \operatorname{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E}) \to \mathcal{K}_N^-(\operatorname{Proj}(\mathcal{E}))$, see Theorem 1.65.*

Proof. Let $f\colon X \to Y$ be a morphism in $\operatorname{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E})$. Denote by $X_\bullet^\bullet$ and $Y_\bullet^\bullet = (Y_\bullet^\bullet, q_\bullet^\bullet, j_\bullet^\bullet)$ the respective projectively resolving admissible N-arrays from Corollary 2.38 with $X_N^\bullet = \rho(X) =: P$ and $Y_N^\bullet = \rho(Y) =: Q$. Consider a lift $g\colon P \to Q$ of f in $C_N(\mathcal{E})$. To prove its uniqueness up to homotopy, we suppose that $f = 0$ and show that g is null-homotopic. To this end, set $h^k := 0$ for $k > 0$, fix $n \leq 0$, and assume for all $k > n$ that $h^k\colon P^k \to Q^{k-N+1}$ is already defined such that $g^k = \sum_{r=0}^{N-1} d_Q^{\{N-r-1\}} h^{k+r} d_P^{\{r\}}$. To construct h^n set $\tilde{g}^n := g^n - \sum_{r=1}^{N-1} d_Q^{\{N-r-1\}} h^{n+r} d_P^{\{r\}}$. We claim that $q_N^n \tilde{g}^n = 0$. If $n = 0$, then $q_N^n \tilde{g}^n = q_N^n g^n = f^{N-1} p_N^n = 0$ as $f = 0$. Otherwise, j_{N-1}^n is a monic, and we verify the equivalent claim $d_Q \tilde{g}^n = j_{N-1}^n q_N^n \tilde{g}^n = 0$:

$$d_Q\left(\sum_{r=1}^{N-1} d_Q^{\{N-r-1\}}h^{n+r}d_P^{\{r\}}\right) = \sum_{r=1}^{N-1} d_Q^{\{N-r\}}h^{n+r}d_P^{\{r\}} = \left(\sum_{r=0}^{N-2} d_Q^{\{N-r-1\}}h^{(n+1)+r}d_P^{\{r\}}\right)d_P$$
$$= \left(g^{n+1} - h^{r+N}d_P^{\{N-1\}}\right)d_P = g^{n+1}d_P = d_Qg^n$$

It follows that $\tilde{g}^n$ factors through the kernel $j_{N-1}^{n-1}\cdots j_1^{n-N+1}$ of q_N^n, that is, $\tilde{g}^n = j_{N-1}^{n-1}\cdots j_1^{n-N+1}\tilde{h}^n$ for some $\tilde{h}^n\colon P^n \to Y_1^{n-N+1}$. Projectivity of P^n yields a lift $h^n\colon P^n \to Q^{n-N+1}$ such that $q_2^{n-N+1}\cdots q_N^{n-N+1}h^n = \tilde{h}^n$. In conclusion, g is null-homotopic as desired:

$$g^n = \tilde{g}^n + \sum_{r=1}^{N-1} d_Q^{\{N-r-1\}}h^{n+r}d_P^{\{r\}} = j_{N-1}^{n-1}\cdots j_1^{n-N+1}q_2^{n-N+1}\cdots q_N^{n-N+1}h^n + \sum_{r=1}^{N-1} d_Q^{\{N-r-1\}}h^{n+r}d_P^{\{r\}}$$
$$= d_Q^{\{N-1\}}h^n + \sum_{r=1}^{N-1} d_Q^{\{N-r-1\}}h^{n+r}d_P^{\{r\}} = \sum_{r=0}^{N-1} d_Q^{\{N-r-1\}}h^{n+r}d_P^{\{r\}}$$

Therefore, up to homotopy the unique lift of the sum of two morphisms equals the sum of the respective unique lifts, and the unique lift of a composition equals the composition of the unique lifts. This makes ρ a full (additive) functor. For faithfulness, suppose that g factors through $I(P)$, see Remark 1.27.(a). Due to Corollary 2.38, $f^r\colon X^r \to Y^r$ factors through the cokernel of $d_{I(P)}^{\{r\}}\colon I(P)^{-N+1} \to I(P)^{r-N+1}$. This cokernel is zero, since $P^k = 0$ for $k > 0$, see Construction 1.51. □

Definition 2.41. Let $f\colon X \to Y$ be a morphism of N-complexes over an exact category $\mathcal{E}$. We call both f and X a **resolution** (of Y) if the cone $C(f)$ is N-acyclic, and **projective** if $X \in C_N(\mathrm{Proj}(\mathcal{E}))$. We call f and Y a **coresolution** (of X) if the cocone $C^*(f)$ is N-acyclic, and **injective** if $Y \in C_N(\mathrm{Inj}(\mathcal{E}))$.

Proposition 2.42. *Let (X, p, i) be a resolving N-array over an exact category $\mathcal{E}$. Then $p^{\{N\}}\colon X_N^\bullet \to X_0^\bullet$ is a resolution, see Remark 2.24. The acyclic N-array (C, q, j) of its cone $C(p^{\{N\}})$ is given by $C_r^n := X_{N-r}^{n+N-r} \oplus \bigoplus_{k=N-r+1}^{N-1} X_N^{n+k}$,*

$$q_r^n = \left(\begin{array}{c|c|c} i & p^{\{r-1\}} & 0 \\ \hline 0 & 0 & E_{r-2} \end{array}\right)\colon C_r^n \twoheadrightarrow C_{r-1}^n,\quad j_r^n = \left(\begin{array}{c|ccc} p & 0 & \cdots & 0 \\ \hline 0 & & E_{r-1} & \\ \hline -i^{\{r\}} & -d_{X_N^\bullet}^{\{r-1\}} & \cdots & -d_{X_N^\bullet} \end{array}\right)\colon C_r^n \rightarrowtail C_{r+1}^{n+1},$$

written in the spirit of Notations 1.20 and 1.57.

Proof. Note that $C_N^n = C(p^{\{N\}})^n$ for all n, see Construction 1.58.(a). For the commutativity of the squares

$$\begin{tikzcd} & C_{r+1}^{n+1} \arrow[dr, "q_{r+1}^{n+1}"] & \\ C_r^n \arrow[ur, "j_r^n"] \arrow[dr, "q_r^n"'] & & C_r^{n+1} \\ & C_{r-1}^n, \arrow[ur, "j_{r-1}^n"'] & \end{tikzcd}$$

we verify the equation $q^{n+1}_{r+1} j^n_r = j^n_{r-1} q^n_r$ in matrix form for all n and r, using the commutativity of the N-array X and $d_{X^\bullet_N} = ip$, see Remark 2.24:

$$\left(\begin{array}{c|c|ccc} ip & p^{\{r\}} & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & E_{r-2} & \\ 0 & 0 & & & \\ \hline -i^{\{r\}} & -d^{\{r-1\}}_{X^\bullet_N} & -d^{\{r-2\}}_{X^\bullet_N} & \cdots & -d_{X^\bullet_N} \end{array}\right) = \left(\begin{array}{c|c|ccc} pi & p^{\{r\}} & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & E_{r-2} & \\ 0 & 0 & & & \\ \hline -i^{\{r\}} & -i^{\{r-1\}}p^{\{r-1\}} & -d^{\{r-2\}}_{X^\bullet_N} & \cdots & -d_{X^\bullet_N} \end{array}\right).$$

For $r = N$, this specializes to $d^n_{C(p^{\{N\}})} = j^n_{N-1} q^n_N$, see Construction 1.58.(a). Due to Proposition 2.30.(b), it remains to prove that the sequence $C^{n-N+r}_r \overset{j^{\{N-r\}}}{\rightarrowtail} C(p^{\{N\}})^n \overset{q^{\{r\}}}{\twoheadrightarrow} C^n_{N-r}$ in $\mathcal{E}$ is short exact for all n and r. In explicit terms, this sequence reads

$$X^n_{N-r} \oplus \bigoplus\nolimits_{k=1}^{r-1} X^{n+k}_N \xrightarrow{j^{\{N-r\}}} X^n_0 \oplus \bigoplus\nolimits_{k=1}^{N-1} X^{n+k}_N \xrightarrow{q^{\{r\}}} X^{n+r}_r \oplus \bigoplus\nolimits_{k=r+1}^{N-1} X^{n+k}_N.$$

Using the commutativity of X again, one computes that

$$j^{\{N-r\}} = \prod_{k=N-1}^{r} j^{n-N+k}_k = \left(\begin{array}{c|ccc} p^{\{N-r\}} & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & E_{r-1} & \\ 0 & & & \\ \hline -i^{\{r\}} & -d^{\{r-1\}}_{X^\bullet_N} & \cdots & -d_{X^\bullet_N} \\ \hline 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{array}\right)$$

and

$$q^{\{r\}} = \prod_{k=r-1}^{0} q^n_{N-k} = \left(\begin{array}{c|c|ccc|c|ccc} i^{\{r\}} & p^{\{N-r\}}d^{\{r-1\}}_{X^\bullet_N} & p^{\{N-r\}}d^{\{r-2\}}_{X^\bullet_N} & \cdots & p^{\{N-r\}}d_{X^\bullet_N} & p^{\{N-r\}} & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & E_{N-r-1} & \\ 0 & 0 & 0 & \cdots & 0 & 0 & & & \end{array}\right).$$

The context for the morphisms in the these matrices is provided by the following excerpt of X, see Remark 1.2:

$$\begin{tikzcd}
X_N^n \arrow[r, "d_{X_N^\bullet}"] \arrow[dr, two heads, "p^{\{r\}}"] & \cdots \arrow[r, "d_{X_N^\bullet}"] & X_N^{n+r} \arrow[r, "d_{X_N^\bullet}"] \arrow[dr, two heads, "p^{\{N-r\}}"] & \cdots \arrow[r, "d_{X_N^\bullet}"] & X_N^{n+N} \\
& X_{N-r}^n \arrow[ur, "i^{\{r\}}"] \arrow[dr, two heads, "p^{\{N-r\}}"] & \diamond & X_r^{n+r} \arrow[ur, "i^{\{N-r\}}"] & \\
& & X_0^n \arrow[ur, "i^{\{r\}}"] & &
\end{tikzcd}$$

By Proposition 1.12.(b), $a' := \begin{pmatrix} p^{\{N-r\}} \\ -i^{\{r\}} \end{pmatrix}$ and $a := \begin{pmatrix} i^{\{r\}} & p^{\{N-r\}} \end{pmatrix}$ define a short exact sequence (a', a) with

$$-ab = \begin{pmatrix} p^{\{N-r\}} d_{X_N^\bullet}^{\{r-1\}} & p^{\{N-r\}} d_{X_N^\bullet}^{\{r-2\}} & \cdots & p^{\{N-r\}} d_{X_N^\bullet} \end{pmatrix} \qquad \text{for} \qquad b := \begin{pmatrix} 0 & \cdots & 0 \\ -d_{X_N^\bullet}^{\{r-1\}} & \cdots & -d_{X_N^\bullet} \end{pmatrix}.$$

Now move the $(r+1)$st row of $j^{\{N-r\}}$ and the $(r+1)$st column of $q^{\{r\}}$ to the respective second position. Then applying Lemma 2.14.(a), followed by Lemma 2.14.(b) with $c = 0$, yields the claim. □

Combining Propositions 2.34 and 2.42 yields

Corollary 2.43. *Every N-complex $X \in C_N^-(\mathcal{E})$ over an exact category $\mathcal{E}$ admits a projective resolution $P \to X$ with $P \in C_N^-(\mathrm{Proj}(\mathcal{E}))$.* □

2.5. **N-syzygies.** Syzygies of 2-complexes consist of single objects of the base exact category $\mathcal{E} = \mathrm{Mor}_0^{\mathrm{m}}(\mathcal{E})$. It seems natural to define the syzygy $\Omega^n X$ of an N-complex X at position n as an element of $\mathrm{Mor}_{N-2}^{\mathrm{m}}(\mathcal{E})$. In this subsection, we prove that these N-syzygies induce a triangle equivalence $\underline{\Omega}^n \colon \underline{\mathrm{APC}}_N(\mathcal{F}) \to \underline{\mathrm{Mor}^{\mathrm{m}}}_{N-2}(\mathcal{F})$ of stable categories, for any Frobenius category $\mathcal{F}$. The required essential surjectivity is given by a complete resolution. This is constructed by patching the one-sided N-array from Corollary 2.38 together with a dual one-sided N-array.

Brightbill and Miemietz [BM24] establish this equivalence in a setup, where $\mathcal{F}$ is a Frobenius subcategory of an Abelian category $\mathcal{E}$ with $\mathrm{Proj}(\mathcal{F}) = \mathrm{Proj}(\mathcal{E})$, see Assumption 3.33. Propositions 2.45, 2.47, and Theorem 2.50 correspond to [BM24, Props. 4.7, 4.3, 4.5, Thm. 4.12].

Definition 2.44. Let $X \in C_N(\mathcal{E})$ be an N-complex over an exact category $\mathcal{E}$ and $n \in \mathbb{Z}$. Suppose that X is acyclic at positions $n-N+1, \ldots, n-1$. We define the **syzygy of X at position n** as the unique object $\Omega^n X := \Omega_{\mathcal{E}}^n X \in \mathrm{Mor}_{N-2}^{\mathrm{m}}(\mathcal{E})$ with $\iota^{n-1}\Omega^n X = \tau^{\leq n-1}\sigma^{\geq n-N+1}X$, see Definition 2.31.(a). In explicit terms, this reads

$$\Omega^n X \colon\ X_1^{n-N+1} \rightarrowtail X_2^{n-N+2} \rightarrowtail \cdots \rightarrowtail X_{N-1}^{n-1},$$

where $X_r^k = C_{(r)}^k(X)$ for all occurring indices k and r. Recall that $X_r^k = Z_{(r)}^n(X)$ for $r \in \{1, \ldots, N-1\}$ if X is acyclic at position n. This gives rise to a functor $C_N^{\infty,\varnothing}(\mathcal{E}) \to \mathrm{Mor}_{N-2}^{\mathrm{m}}(\mathcal{E})$.

To show exactness of Ω^n, the proof of [BM24, Prop. 4.7] uses the snake lemma in an Abelian category $\mathcal{E}$, which is not available in general exact categories. However, Lemma 2.6 serves as a replacement.

Proposition 2.45. *Let $\mathcal{E}$ be an exact category. The functor $\Omega^n : C_N^{\infty,\varnothing}(\mathcal{E}) \to \mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E})$ is exact for all $n \in \mathbb{Z}$.*

Proof. Due to the termwise exact structure of $\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E})$, exactness of Ω^n is equivalent to exactness of $C^{n-r}_{(N-r)}$, for all $r \in \{1, \dots, N-1\}$. Hence, the claim follows from Lemma 2.6 □

Definition 2.46. Let $\mathcal{E}$ be an exact category and $X \in \mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E})$. A **complete resolution** of X is an object $P \in \mathrm{TAPC}_N(\mathcal{E})$ with $\Omega^1(P) = X$.

Over a Frobenius category $\mathcal{F}$, $\mathrm{TAPC}_N(\mathcal{F}) = \mathrm{APC}_N(\mathcal{F})$, see Notation 2.19, and Proposition 2.47 establishes the existence of complete resolutions. The use of Proposition 1.14 in Corollary 2.38 allows us to extend the arguments of [BM24, Prop. 4.5] from the case where $\mathcal{E}$ is Abelian.

Proposition 2.47. *If $\mathcal{F}$ is a Frobenius category, then Ω^n restricts to an essentially surjective and full functor $\mathrm{APC}_N(\mathcal{F}) \to \mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{F})$, for all $n \in \mathbb{Z}$.*

Proof. We may assume that $n = 1$. For essential surjectivity, consider an arbitrary object

$$X^1 \rightarrowtail X^2 \rightarrowtail \cdots \rightarrowtail X^{N-2} \rightarrowtail X^{N-1}$$

of $\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{F})$. In view of Theorem 2.26, it suffices to construct an acyclic N-array $X^{\bullet}_{\bullet} \in A_N(\mathcal{F})$ with $X^k_N \in \mathrm{Proj}(\mathcal{F}) = \mathrm{Inj}(\mathcal{F})$, for all $k \in \mathbb{Z}$, and $X_r^{-N+r+1} = X^r$, for all $r \in \{1, \dots, N-1\}$. Corollary 2.38 yields the "left half" of the desired array. To complete the array we apply the dual construction to the element

$$X_{N-1}^{-N+2} \twoheadrightarrow X_{N-2}^{-N+2} \twoheadrightarrow \cdots \twoheadrightarrow X_2^{-N+2} \twoheadrightarrow X^1$$

of $\mathrm{Mor}^{\mathrm{e}}_{N-2}(\mathcal{F})$ using the already constructed bicartesian squares and projective-injectives. In view of 2.26, fullness follows from Remark 2.37 and its dual. □

Remark 2.48. For an object A of an exact category $\mathcal{E}$,

$$\Omega^n \mu_N^s(A) = \begin{cases} 0, & \text{if } n > s \text{ or } n \leq s - N + 1, \\ \mu_{N+n-s-1}(A), & \text{otherwise,} \end{cases}$$

which is an object of $\mathrm{Mor}^{\mathrm{sm}}_{N-2}(\mathcal{E})$, see Remark 1.62 and Notation 1.70.

Lemma 2.49. *Let $\mathcal{E}$ be an exact category and $Q \in \mathrm{APC}_N(\mathcal{E})$. Then $\Omega^n P(Q) \in \mathrm{Proj}(\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E}))$ for all $n \in \mathbb{Z}$, see Construction 1.51. In particular, $\Omega^n = \Omega^n_{\mathcal{E}}$ induces a unique functor $\underline{\Omega}^n = \underline{\Omega}^n_{\mathcal{E}}$ of stable categories:*

$$\begin{tikzcd}
\mathrm{APC}_N(\mathcal{E}) \arrow[r, "\Omega^n_{\mathcal{E}}"] \arrow[d] & \mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E}) \arrow[d] \\
\underline{\mathrm{APC}}_N(\mathcal{E}) \arrow[r, dashed, "\underline{\Omega}^n_{\mathcal{E}}"] & \underline{\mathrm{Mor}^{\mathrm{m}}}_{N-2}(\mathcal{E})
\end{tikzcd}$$

Proof. Using Remark 2.48, we find that $\Omega^n P(Q) \in \mathrm{Mor}^{\mathrm{sm}}_{N-2}(\mathrm{Proj}(\mathcal{E})) = \mathrm{Proj}\left(\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E})\right)$, see Construction 1.51 and Theorem 1.71.(b). The particular claim follows using Remark 1.27.(a) and Construction 1.51. $\square$

Brightbill and Miemietz show in [BM24, §4.2] that $\underline{\Omega}^n_{\mathcal{F}}$ is faithful in their setup. They construct a null-homotopy for a morphism $f : P \to Q$ in $\mathrm{APC}_N(\mathcal{F})$ by lifting a factorization of $\Omega^n(f)$ through an element of $\mathrm{Proj}(\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{F}))$. The quotient objects in their proof, formed in the ambient Abelian category, occur in the acyclic N-arrays of P and Q, see Theorem 2.26. So, their argument works almost verbatim in our setup Combining this with Propositions 2.45 and 2.47, Lemma 2.49 and Proposition 1.35 yields

Theorem 2.50. *If $\mathcal{F}$ is Frobenius category, then the functor $\underline{\Omega}^n\colon \underline{\mathrm{APC}}_N(\mathcal{F}) \to \underline{\mathrm{Mor}^{\mathrm{m}}}_{N-2}(\mathcal{F})$ is a triangle equivalence for all $n \in \mathbb{Z}$.* $\square$

3. Stabilized N-derived categories

In this section, we complete the diagram in Theorem A by the (stabilized) N-derived category and establish the two remaining (stabilized) functors $\underline{\iota}^0$ and $\underline{\tau}^{\leq 0}$. Along the way we provide analogues of known results on 2-derived categories for general N. The proof that the stabilized truncation $\underline{\tau}^{\leq 0}$ is a triangle equivalence occupies a substantial part of this section.

3.1. N-derived categories.

In this subsection, we consider the N-derived category $\mathcal{D}_N(\mathcal{E})$ of an exact idempotent complete category $\mathcal{E}$ and its subcategories $\mathcal{D}^{\#,\natural}_N(\mathcal{E})$ given by boundedness conditions. These are defined as Verdier quotients of the corresponding homotopy categories $\mathcal{K}^{\#,\natural}_N(\mathcal{E})$ by their respective triangulated subcategory $\mathcal{K}^{\#,\varnothing}_N(\mathcal{E})$ of acyclic N-complexes. We establish several fundamental properties known from the classical case, where $N = 2$ and $\mathcal{E}$ is Abelian. The case $N = 2$ was considered by Keller [Kel96], the Abelian case by Iyama, Kato and Miyachi [IKM17]. Like Keller, we impose idempotent completeness of $\mathcal{E}$ to ensure that N-acyclicity is preserved under homotopy equivalence. Notably, we obtain canonical inclusions $\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E}) \subseteq \mathcal{D}^b_N(\mathcal{E}) \subseteq \mathcal{D}^{\#}_N(\mathcal{E}) \subseteq \mathcal{D}_N(\mathcal{E})$ as triangulated subcategories and triangle equivalences $\mathcal{D}^-_N(\mathcal{E}) \simeq \mathcal{K}^-_N(\mathrm{Proj}(\mathcal{E}))$ and $\mathcal{D}^b_N(\mathcal{E}) \simeq \mathcal{D}^{-,b}_N(\mathcal{E}) \simeq \mathcal{K}^{-,b_{\mathcal{E}}}_N(\mathrm{Proj}(\mathcal{E}))$. Finally, we relate syzygies to truncations of N-complexes in $\mathcal{D}^{-,b}_N(\mathcal{E})$, a key ingredient for the proof of Theorem A.

Bühler proves Proposition 3.1, proposed by Keller, for $N = 2$, see [Büh10, Prop. 10.9]. We adapt his arguments for general N.

Proposition 3.1. *The following conditions are equivalent for any exact category $\mathcal{E}$:*

1. *Every null-homotopic N-complex in $C_N(\mathcal{E})$ is acyclic.*
2. *The category $\mathcal{E}$ is idempotent complete.*
3. *If a morphism $X \to Y$ in $\mathcal{K}_N(\mathcal{E})$ admits a left-inverse and Y is acyclic at position $n \in \mathbb{Z}$, then so is X.*

In particular, the full, but not essential, image of $\mathcal{K}^{\infty,\varnothing}_N(\mathcal{E})$ in $\mathcal{K}_N(\mathcal{E})$ is thick if and only if $\mathcal{E}$ is idempotent complete.

Proof.

(3) ⇒ (1) If $X \in C_N(\mathcal{E})$ is null-homotopic, then there is an isomorphism $X \to 0$ in $\mathcal{K}_N(\mathcal{E})$, see Definition 1.66. As the zero N-complex is acyclic, (3) implies that X is so as well.

(1) ⇒ (2) For an idempotent $e\colon A \to A$ in $\mathcal{E}$, consider the N-complex

$$X\colon\ \cdots \xrightarrow{\mathrm{id}} X^0 \xrightarrow{e} X^1 \xrightarrow{1-e} X^2 \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} X^N \xrightarrow{e} X^{N+1} \xrightarrow{1-e} X^{N+2} \xrightarrow{\mathrm{id}} \cdots$$

where $X^i = A$ for all $i \in \mathbb{Z}$. One can see that the morphism id_X is null-homotopic with homotopy h defined by $h^i = \mathrm{id}_A$. So, X is acyclic by (1) and e splits with kernel $Z^0_{(1)}(X)$, see Definition 1.75.

(2) ⇒ (3) Let $s\colon X \to Y$ be a morphism in $C_N(\mathcal{E})$ with left-inverse $r\colon Y \to X$ in $\mathcal{K}_N(\mathcal{E})$. This means that $rs - \mathrm{id}_X$ factors in $C_N(\mathcal{E})$ through $i = i_X\colon X \rightarrowtail I(X)$, see Remark 1.27.(a) and Construction 1.51:

$$\begin{tikzcd}
X \arrow[rr, "rs - \mathrm{id}_X"] \arrow[dr, tail, "i"'] & & X \\
& I(X) \arrow[ur, "f"'] &
\end{tikzcd}$$

Equivalently, the morphism $\begin{pmatrix} i \\ s \end{pmatrix} : X \to I(X) \oplus Y$, has the left-inverse $\begin{pmatrix} -f & r \end{pmatrix} : I(X) \oplus Y \to X$ in $C_N(\mathcal{E})$. Suppose that Y and hence $I(X) \oplus Y$ is acyclic at position n, see Remark 2.10.(b) and Proposition 1.8. Replacing s by $\begin{pmatrix} i \\ s \end{pmatrix}$ we may assume that s has a left inverse $r\colon Y \to X$ in $C_N(\mathcal{E})$. Due to (2), Lemma 3.2 applies to the idempotent $e := sr : Y \to Y$ and shows that $X \cong_{C_N(\mathcal{E})} eY$ is acyclic at position n, see Remark 1.76.

The particular claim follows since thickness follows from (3) and implies (1). □

Lemma 3.2. *Consider an idempotent $e\colon X \to X$ of an N-complex X over an exact idempotent complete category $\mathcal{E}$. If X is acyclic at position $n \in \mathbb{Z}$, then so is eX, see Proposition 1.81.(a).*

Proof. By applying the contraction functors γ^n_r for $r \in \{1, \ldots, N-1\}$, we may assume that $N = 2$. There are induced idempotents e^n on $Z^n = Z^n(X)$ and $C^n = C^n(X)$, see Remark 1.79, and the diagram

$$\begin{tikzcd}
X^{n-1} \arrow[rr, "d^{n-1}", "\circ" marking] \arrow[dr, two heads] \arrow[ddd, "e^{n-1}"] & & X^n \arrow[rr, "d^n", "\circ" marking] \arrow[dr, two heads] \arrow[ddd, "e^n"] & & X^{n+1} \arrow[ddd, "e^{n+1}"] \\
& Z^n \arrow[ur, tail] \arrow[d, dashed, "e^n"] & & C^n \arrow[ur, tail] \arrow[d, dashed, "e^n"] & \\
& Z^n \arrow[dr, tail] & & C^n \arrow[dr, tail] & \\
X^{n-1} \arrow[rr, "d^{n-1}", "\circ" marking] \arrow[ur, two heads] & & X^n \arrow[rr, "d^n", "\circ" marking] \arrow[ur, two heads] & & X^{n+1}
\end{tikzcd}$$

commutes, see Remark 2.7. Applying Lemma 1.80 and its dual yields a commutative diagram

$$\begin{tikzcd}
 & & & e^nC^n \arrow[dr, dashed, tail] \arrow[dd, dashed, tail] & \\
e^{n-1}X^{n-1} \arrow[rr] \arrow[dr, dashed, two heads] \arrow[dd, tail] & & e^nX^n \arrow[rr] \arrow[ur, dashed, two heads] \arrow[dd, tail] & & e^{n+1}X^{n+1} \arrow[dd, tail] \\
 & e^nZ^n \arrow[ur, dashed, tail] \arrow[dd, dashed, tail] & & C^n \arrow[dr, tail] \arrow[dd, dashed, two heads] & \\
X^{n-1} \arrow[rr] \arrow[dr, two heads] \arrow[dd, two heads] & & X^n \arrow[rr] \arrow[ur, two heads] \arrow[dd, two heads] & & X^{n+1} \arrow[dd, two heads] \\
 & Z^n \arrow[ur, tail] \arrow[dd, dashed, two heads] & & (1-e^n)C^n \arrow[dr, dashed, tail] & \\
(1-e^{n-1})X^{n-1} \arrow[rr] \arrow[dr, dashed, two heads] & & (1-e^n)X^n \arrow[rr] \arrow[ur, dashed, two heads] & & (1-e^{n+1})X^{n+1} \\
 & (1-e^n)Z^n \arrow[ur, dashed, tail] & & &
\end{tikzcd}$$

such that all sequences $\bullet \rightarrowtail \bullet \twoheadrightarrow \bullet$ are short exact. This implies the claim. □

Corollary 3.3. *If* $f\colon X \to Y$ *is a resolution over an exact idempotent complete category* $\mathcal{E}$ *with* $X \in C_N^-(\mathrm{Proj}(\mathcal{E}))$ *and* $Y \in C_N^{-,b}(\mathcal{E})$*, then* $X \in C_N^{-,b_{\mathcal{E}}}(\mathrm{Proj}(\mathcal{E}))$.

Proof. As $\mathcal{K}_N^{-,b}(\mathcal{E})$ is a triangulated subcategory of $\mathcal{K}_N^-(\mathcal{E})$, see Theorem 2.20.(c), and $C(f) \in C_N^{-,\varnothing}(\mathcal{E}) \subseteq C_N^{-,b}(\mathcal{E})$, this implies that X is isomorphic in $\mathcal{K}_N(\mathcal{E})$ to a complex in $C_N^{-,b}(\mathcal{E})$ and hence lies in $C_N^{-,b_{\mathcal{E}}}(\mathrm{Proj}(\mathcal{E}))$ by Proposition 3.1.(3). □

Corollary 3.4. *Let* $\mathcal{E}$ *be an exact idempotent complete category and* $\#, \#' \in\in \{\infty, -, b\}$ *and* $\natural, \natural' \in \{\infty, -, b, \varnothing\}$ *with* $(\#', \natural') \leq (\#, \natural)$*. Then* $\mathcal{K}_N^{\#',\natural'}(\mathcal{E}) = \mathcal{K}_N^{\#',\natural}(\mathcal{E}) \cap \mathcal{K}_N^{\#,\natural'}(\mathcal{E})$ *as subcategories of* $\mathcal{K}_N^{\#,\natural}(\mathcal{E})$.

Proof. Let $X \in \mathcal{K}_N^{\#',\natural}(\mathcal{E}) \cap \mathcal{K}_N^{\#,\natural'}(\mathcal{E})$. This means that X is isomorphic in $\mathcal{K}_N(\mathcal{E})$ to both an object $Z \in C_N^{\#',\natural}(\mathcal{E})$ and $Y \in C_N^{\#,\natural'}(\mathcal{E})$. Proposition 3.1.(3) applied to $Z \cong_{\mathcal{K}_N(\mathcal{E})} Y$ yields that $Z \in C_N^{\#',\natural}(\mathcal{E}) \cap C_N^{\infty,\natural'}(\mathcal{E}) = C_N^{\#',\natural'}(\mathcal{E})$. So, $X \cong_{\mathcal{K}_N(\mathcal{E})} Z$ lies in the subcategory $\mathcal{K}_N^{\#',\natural'}(\mathcal{E})$ of $\mathcal{K}_N(\mathcal{E})$. The converse inclusion is obvious. □

Definition 3.5. The ***N*-derived categories** of an exact idempotent complete category $\mathcal{E}$ are defined as the Verdier quotients $\mathcal{D}_N^{\#,\natural}(\mathcal{E}) := \mathcal{K}_N^{\#,\natural}(\mathcal{E})/\mathcal{K}_N^{\#,\varnothing}(\mathcal{E})$, for $\# \in \{\infty, +, -, b\}$ and $\natural \in \{\infty, +, -, b\}$. A morphism in $C_N(\mathcal{E})$ is called an **(*N*-)quasi-isomorphism** if it is an isomorphism in $\mathcal{D}_N(\mathcal{E})$.

Lemma 3.6. *Let* $\mathcal{E}'$ *be an exact subcategory of an exact idempotent complete category* $\mathcal{E}$*. Then there is a canonical triangle functor* $\mathcal{D}_N^{\#,\natural}(\mathcal{E}') \to \mathcal{D}_N^{\#,\natural}(\mathcal{E})$*, for* $\# \in \{\infty, +, -, b\}$ *and* $\natural \in \{\infty, +, -, b\}$.

Proof. This follows from the universal property of the Verdier quotient, since the fully faithful functor triangle $\mathcal{K}_N^{\#,\natural}(\mathcal{E}') \to \mathcal{K}_N^{\#,\natural}(\mathcal{E})$ from Proposition 2.21 sends $\mathcal{K}_N^{\#,\varnothing}(\mathcal{E}')$ to $\mathcal{K}_N^{\#,\varnothing}(\mathcal{E})$. □

Due to the particular statement of Proposition 3.1, Proposition 3.7 is a special case of a general fact on Verdier quotients, see [Nee01, Prop. 2.1.35].

Proposition 3.7. *Over an exact idempotent complete category, resolutions of N-complexes agree with N-quasi-isomorphisms.* □

Henrard and van Roosmalen [HR20] consider 2-complexes over (more general) deflation-exact categories. We reduce the claim of Lemma 3.8 to their argument by contraction.

Lemma 3.8. *Over an exact idempotent complete category $\mathcal{E}$, any term*wise *short exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $C_N(\mathcal{E})$, see Remark 1.22, induces a distinguished triangle in $\mathcal{D}_N(\mathcal{E})$:*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow \Sigma X$$

Proof. The morphism f fits into the distinguished triangle $T(f)$

$$X \xrightarrow{f} Y \longrightarrow C(f) \xrightarrow{w} \Sigma X$$

in $\mathcal{K}_N(\mathcal{E})$ and hence in $\mathcal{D}_N(\mathcal{E})$, see Construction 1.31. Since $gf = 0$, there is a morphism

$$\begin{tikzcd}
X \arrow[r, tail, "\begin{pmatrix} i_X \\ f \end{pmatrix}"] \arrow[d, "\mathrm{id}_X"] & I(X) \oplus Y \arrow[r, two heads] \arrow[d, two heads, "\begin{pmatrix} 0 & \mathrm{id}_Y \end{pmatrix}"] & C(f) \arrow[d, dashed, "h_f"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd} \tag{3.1}$$

of term*wise* short exact sequences in $\mathcal{E}$, see Constructions 1.31 and 1.51. We claim that the cone $C(h_f)$ is N-acyclic. Then h_f is an N-quasi-isomorphism, see Proposition 3.7, and the isomorphism

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & Y \arrow[r] \arrow[d, "\mathrm{id}"] & C(f) \arrow[r, "w"] \arrow[d, "h_f"] & \Sigma X \arrow[d, "\mathrm{id}"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, dashed, "w \circ h_f^{-1}"] & \Sigma X
\end{tikzcd}$$

of candidate triangles in $\mathcal{D}_N(\mathcal{E})$ yields the claim. Henrard and van Roosmalen prove this statement in [HR20, Prop. 3.23] for $N = 2$. In particular, $C(h_{\gamma f})$ is 2-acyclic, where $\gamma := \gamma_r^n$ for arbitrary $n \in \mathbb{Z}$ and $r \in \{1, \ldots, N-1\}$. We show that $C(h_{\gamma f}) \cong C(\gamma h_f)$ in $\mathcal{K}_2(\mathcal{E})$. Then 2-acyclicity of $\gamma C(h_f)$ and thus N-acyclicity of $C(h_f)$ follow from Lemma 2.16.(b) and Proposition 3.1.(3).

To this end, we apply γ to (3.1) and combine the result with (2.1), see Construction 2.15. This leads to a commutative diagram

$$\begin{tikzcd}
 & \gamma X \arrow[rr, tail] \arrow[dl, "\mathrm{id}"'] \arrow[dd, "\mathrm{id}"] & & I(\gamma X) \oplus \gamma Y \arrow[rr, two heads] \arrow[dl, tail] \arrow[dd, two heads] & & C(\gamma f) \arrow[dl, tail, "c_X"'] \arrow[dd, "h_{\gamma f}"] \\
\gamma X \arrow[rr, tail] \arrow[dr, "\mathrm{id}"'] & & \gamma I(X) \oplus \gamma Y \arrow[rr, two heads] \arrow[dr] & & \gamma C(f) \arrow[dr, "\gamma h_f"] & \\
 & \gamma X \arrow[rr, "\gamma f"] & & \gamma Y \arrow[rr, "\gamma g"] & & \gamma Z
\end{tikzcd}$$

in $C_2(\mathcal{E})$ with termwise short exact rows. Indeed, $h_{\gamma f} = \gamma h_f \circ c_X$ by the universal property of cokernel. The isomorphism c_X in $\mathcal{K}_2(\mathcal{E})$ from Lemma 2.16.(a) then fits into an isomorphism

$$\begin{tikzcd}
C(\gamma f) \arrow[r, "h_{\gamma f}"] \arrow[d, "c_X"] & \gamma Z \arrow[r] \arrow[d, "\mathrm{id}"] & C(h_{\gamma f}) \arrow[r] \arrow[d, dashed, "\cong"] & \Sigma C(\gamma f) \arrow[d, "\Sigma c_X"] \\
\gamma C(f) \arrow[r, "\gamma h_f"] & \gamma Z \arrow[r] & C(\gamma h_f) \arrow[r] & \Sigma(\gamma C(f))
\end{tikzcd}$$

of distinguished triangles in $\mathcal{K}_2(\mathcal{E})$, which yields the claim. □

We now prove Theorems B and C.

Theorem 3.9. *For an exact idempotent complete category $\mathcal{E}$, there is a diagram of canonical fully faithful, triangle functors and equivalences:*

$$\begin{tikzcd}
 & & \mathcal{D}_N^{+}(\mathcal{E}) \arrow[dr, "\simeq"] & & \\
 & \mathcal{D}_N^{+,b}(\mathcal{E}) \arrow[ur] \arrow[dr, "\simeq"] & & \mathcal{D}_N^{\infty,+}(\mathcal{E}) \arrow[dr] & \\
\mathcal{D}_N^{b}(\mathcal{E}) \arrow[ur, "\simeq"] \arrow[dr, "\simeq"'] & & \mathcal{D}_N^{\infty,b}(\mathcal{E}) \arrow[ur] \arrow[dr] & & \mathcal{D}_N(\mathcal{E}) \\
 & \mathcal{D}_N^{-,b}(\mathcal{E}) \arrow[ur, "\simeq"'] \arrow[dr] & & \mathcal{D}_N^{\infty,-}(\mathcal{E}) \arrow[ur] & \\
 & & \mathcal{D}_N^{-}(\mathcal{E}) \arrow[ur, "\simeq"'] & &
\end{tikzcd}$$

Proof. By symmetry, it suffices to consider the arrows pointing upwards. Let $\#, \#', \natural, \natural' \in \{\infty, -, b\}$ with $(\#', \natural') \leq (\#, \natural)$. We use Theorem 1.87.(c) to establish the functor $\mathcal{D}_N^{\#',\natural'}(\mathcal{E}) \to \mathcal{D}_N^{\#,\natural}(\mathcal{E})$. By Theorem 2.20.(c) and Corollary 3.4, $\mathcal{U} := \mathcal{K}_N^{\#',\natural'}(\mathcal{E})$, $\mathcal{V} := \mathcal{K}_N^{\#,\varnothing}(\mathcal{E})$, and $\mathcal{U} \cap \mathcal{V} = \mathcal{K}_N^{\#',\varnothing}(\mathcal{E})$ are triangulated subcategories of $\mathcal{T} := \mathcal{K}_N^{\#,\natural}(\mathcal{E})$. Condition (3) in Theorem 1.87 holds trivially if $\#' = \#$, since $\mathcal{U} \cap \mathcal{V} = \mathcal{V}$. For the functor $\mathcal{D}_N^{-}(\mathcal{E}) \to \mathcal{D}_N^{\infty,-}(\mathcal{E})$, consider a morphism $X \to Y$ in $\mathcal{T}$ with $X \in C_N^{-}(\mathcal{E})$ and $Y \in C_N^{\infty,\varnothing}(\mathcal{E})$. For any sufficiently large $n \in \mathbb{Z}$, it factors through the canonical morphism $\sigma^{\leq n} Y \to Y$ with $\sigma^{\leq n} Y \in C_N^{-,\varnothing}(\mathcal{E})$ by Corollary 2.32. To show its essential surjectivity, let $X \in C_N^{\infty,-}(\mathcal{E})$. Due to Corollary 2.32 and Lemma 3.8, for sufficiently large $n \in \mathbb{Z}$, there is a distinguished triangle

$$\sigma^{\leq n} X \longrightarrow X \longrightarrow \sigma^{\geq n-N+2} X \longrightarrow \Sigma \sigma^{\leq n} X$$

in $\mathcal{D}_N^{\infty,-}(\mathcal{E})$ with $\sigma^{\geq n-N+2} X = 0 \in \mathcal{D}_N^{-,b}(\mathcal{E})$ and hence $X \cong \sigma^{\leq n} X \in \mathcal{D}_N^{-}(\mathcal{E})$. The preceding arguments restrict to $\mathcal{D}_N^{-,b}(\mathcal{E}) \to \mathcal{D}_N^{\infty,b}(\mathcal{E})$ and $\mathcal{D}_N^{b}(\mathcal{E}) \to \mathcal{D}_N^{+,b}(\mathcal{E})$. □

Lemma 3.10. *Let $X, Y \in C_N(\mathcal{E})$ be N-complexes over an exact idempotent complete category $\mathcal{E}$. Then the canonical functor $\mathcal{K}_N(\mathcal{E}) \to \mathcal{D}_N(\mathcal{E})$ induces an isomorphism $\mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(X, Y) \cong \mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(X, Y)$ of Abelian groups if $X \in C_N^-(\mathrm{Proj}(\mathcal{E}))$ or $Y \in C_N^+(\mathrm{Inj}(\mathcal{E}))$.*

Proof. If $\mathcal{E}$ is Abelian, $\mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(X, \mathcal{K}_N^{\infty,\varnothing}(\mathcal{E})) = 0$ if $X \in C_N^-(\mathrm{Proj}(\mathcal{E}))$ and $\mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(\mathcal{K}_N^{\infty,\varnothing}(\mathcal{E}), Y) = 0$ if $Y \in C_N^+(\mathrm{Inj}(\mathcal{E}))$ by [IKM17, Lem. 3.3]. However, the proof works in general without the use of homology. Then Verdier's criterion [Ver77, Ch. I, §2, n$^{\mathrm{o}}$ 5, 5-3 Prop.] yields the claim. □

Theorem 3.11. *Let $\mathcal{E}$ be an exact idempotent complete category with enough projectives.*

(a) The pair $(\mathcal{K}_N^-(\mathrm{Proj}(\mathcal{E})), \mathcal{K}_N^{-,\varnothing}(\mathcal{E}))$ is a semiorthogonal decomposition of $\mathcal{K}_N^-(\mathcal{E})$, which gives rise to a triangle equivalence $\mathcal{D}_N^-(\mathcal{E}) \simeq \mathcal{K}_N^-(\mathrm{Proj}(\mathcal{E}))$.

(b) The pair $(\mathcal{K}_N^{-,b\mathcal{E}}(\mathrm{Proj}(\mathcal{E})), \mathcal{K}_N^{-,\varnothing}(\mathcal{E}))$ is a semiorthogonal decomposition of $\mathcal{K}_N^{-,b}(\mathcal{E})$, which gives rise to a triangle equivalence $\mathcal{D}_N^b(\mathcal{E}) \simeq \mathcal{K}_N^{-,b\mathcal{E}}(\mathrm{Proj}(\mathcal{E}))$.

Proof. Lemma 3.10 yields condition (a) in Definition 1.83, condition (b) follows from the standard triangle of a resolution, see Corollaries 2.43 and 3.3. The triangle equivalences are then due to Proposition 1.84 using the triangle equivalence $\mathcal{D}_N^b(\mathcal{E}) \simeq \mathcal{D}_N^{-,b}(\mathcal{E})$ from Theorem 3.9. □

Remark 3.12. In view of Remark 1.86, the triangle equivalences in Theorem 3.11 send an N-complex X to P for a projective resolution $P \to X$, and lift morphisms.

Proposition 3.13. *Let $\mathcal{E}$ be an exact idempotent complete category. Then the composed functor $\mathrm{Mor}_{N-2}^{\mathrm{m}}(\mathcal{E}) \xrightarrow{\iota^n} C_N^b(\mathcal{E}) \to \mathcal{D}_N^b(\mathcal{E})$, see Notation 1.69, is fully faithful and also denoted by ι^n.*

Proof. We may assume that $n = 0$. Due to Remark 3.12, postcomposing ι^0 with the equivalence $\mathcal{D}_N^b(\mathcal{E}) \to \mathcal{K}_N^{-,b\mathcal{E}}(\mathrm{Proj}(\mathcal{E}))$ and the fully faithful functor $\mathcal{K}_N^{-,b\mathcal{E}}(\mathrm{Proj}(\mathcal{E})) \to \mathcal{K}_N^-(\mathrm{Proj}(\mathcal{E}))$, see Theorems 2.20.(c) and 3.11.(b), yields the functor ρ from Proposition 2.40. Then ι^0 is fully faithful since ρ is so. □

Definition 3.14. For an additive category $\mathcal{A}$, the **(left hard) truncation** $\tau^{\leq n}$ at $n \in \mathbb{Z}$ is the exact functor $C_N(\mathcal{A}) \to C_N^-(\mathcal{A})$ sending $X \in C_N(\mathcal{A})$ to

$$\tau^{\leq n}X\colon \qquad \cdots \xrightarrow{d_X^{n-3}} X^{n-2} \xrightarrow{d_X^{n-2}} X^{n-1} \xrightarrow{d_X^{n-1}} X^n \longrightarrow 0 \longrightarrow \cdots .$$

The **(right hard) truncation** $\tau^{\geq n}$ is defined analogously.

Remark 3.15. For any N-complex $X \in C_N(\mathcal{E})$ over an exact category $\mathcal{E}$ and $n \in \mathbb{Z}$, there is a termsplit short exact sequence $\tau^{\geq n}X \rightarrowtail X \twoheadrightarrow \tau^{\leq n-1}X$. It induces a distinguished triangle

$$\tau^{\geq n}X \longrightarrow X \longrightarrow \tau^{\leq n-1} \longrightarrow \Sigma\tau^{\geq n}X$$

in $\mathcal{K}_N(\mathcal{E})$, see Lemma 1.33, and hence in $\mathcal{D}_N(\mathcal{E})$ if $\mathcal{E}$ is idempotent complete.

Reversing the construction in Corollary 2.38, we relate truncations and syzygies of acyclic N-complexes. This is generalizes the quasi-isomorphism between an object and its resolution, known from the case $N = 2$.

Lemma 3.16. *Let $X \in C_N(\mathcal{E})$ be an N-complex over an exact idempotent complete category $\mathcal{E}$. Suppose that X is acyclic at all positions up to $n \in \mathbb{Z}$. Then the canonical morphism*

$$\begin{tikzcd}
\tau^{\leq n}X\colon \arrow[d] & \cdots \arrow[r] & X^{n-N+1} \arrow[r, "d_X"] \arrow[d, two heads] & X^{n-N+2} \arrow[r, "d_X"] \arrow[d, two heads] & \cdots \arrow[r, "d_X"] & X^{n-1} \arrow[r, "d_X"] \arrow[d, two heads] & X^n \arrow[d, two heads] \\
\iota^n\Omega^{n+1}X\colon & \cdots \arrow[r] & 0 \arrow[r] & C^{n-N+2}_{(1)}(X) \arrow[r, tail] & \cdots \arrow[r, tail] & C^{n-1}_{(N-2)}(X) \arrow[r, tail] & C^{n}_{(N-1)}(X)
\end{tikzcd}$$

is a resolution. In particular, there is an isomorphism of functors $\tau^{\leq n} \cong \iota^n \circ \Omega^{n+1}\colon \operatorname{APC}_N(\mathcal{E}) \to \mathcal{D}_N^{-,b}(\mathcal{E})$.

Proof. Patch all diagrams that can be obtained from Proposition 2.30 as in the proof of Theorem 2.26. Then modify the resulting diagram at the right end as follows, and extend by zero to create a resolving N-array $X^\bullet_\bullet$ of $X^\bullet_0 = \iota^n\Omega^{n+1}X$ with $X^\bullet_N = \tau^{\leq n}X$:

$$\begin{tikzcd}[column sep=tiny]
\cdots \arrow[r] & X^{n-N+2} \arrow[rr] \arrow[dr, two heads] & & X^{n-N+3} \arrow[r, dash] & \cdots \arrow[r] & X^{n-1} \arrow[rr] \arrow[dr, two heads] & & X^n \arrow[dr, two heads] & & & & & \\
\iddots \arrow[ur, tail] & & C^{n-N+2}_{(N-1)} \arrow[ur, tail] \arrow[dr, two heads] & & & & C^{n-1}_{(N-1)} \arrow[ur, tail] \arrow[dr, two heads] & & C^{n}_{(N-1)} \arrow[dr, equal] & & & & \\
& \iddots \arrow[ur, tail] & & \ddots \arrow[dr, two heads] & & \iddots \arrow[ur, tail] & & C^{n-1}_{(N-2)} \arrow[ur, tail] \arrow[dr, equal] & & C^{n}_{(N-1)} \arrow[dr, equal] & & & \\
& & \ddots \arrow[dr, two heads] & & C^{n-N+2}_{(2)} \arrow[ur, tail] \arrow[dr, two heads] & & \iddots \arrow[ur, tail] & & C^{n-1}_{(N-2)} \arrow[ur, tail] \arrow[dr, equal] & & \ddots \arrow[dr, equal] & & \\
& & & C^{n-N+1}_{(1)} \arrow[ur, tail] \arrow[dr, two heads] & & C^{n-N+2}_{(1)} \arrow[ur, tail] \arrow[dr, equal] & & \iddots \arrow[ur, tail] & & \ddots \arrow[dr, equal] & & C^{n}_{(N-1)} \arrow[dr, equal] & \\
\cdots \arrow[rr] & & 0 \arrow[rr] \arrow[ur, tail] & & 0 \arrow[rr, tail] \arrow[ur, tail] & & C^{n-N+2}_{(1)} \arrow[rr, tail] \arrow[ur, tail] & & \cdots \arrow[rr, tail] & & C^{n-1}_{(N-2)} \arrow[rr, tail] \arrow[ur, tail] & & C^{n}_{(N-1)}
\end{tikzcd}$$

Then Proposition 2.42 and yield the desired resolution. Naturality is clear. □

3.2. **Perfect N-complexes.** In this subsection, we consider the subcategory $\mathcal{D}_N^{\mathrm{perf}}(\mathcal{E})$ of perfect N-complexes of $\mathcal{D}^b_N(\mathcal{E})$. We characterize it by means of an Ext condition due to Buchweitz in the classical case. The corresponding Verdier quotient is the N-singularity category $\underline{\mathcal{D}}^b_N(\mathcal{E})$. We show that its objects are obtained by embedding $\underline{\mathrm{Mor}^{\mathrm{m}}}_{N-2}(\mathcal{E})$ at various positions, generalizing a statement of Orlov's (Lemma 3.25).

If $\mathcal{E}$ is idempotent complete, then so are its subcategories $\operatorname{Proj}(\mathcal{E})$ and $\operatorname{Inj}(\mathcal{E})$, see Remark 1.25. This allows for the following

Definition 3.17. The category of **perfect** N-complexes over an exact idempotent complete category $\mathcal{E}$ is defined as $\mathcal{D}_N^{\mathrm{perf}}(\mathcal{E}) := \mathcal{D}^b_N(\operatorname{Proj}(\mathcal{E}))$.

Lemma 3.18. *Let $\mathcal{E}$ be an exact idempotent complete category with enough projectives.*

(a) The canonical triangle functor $\mathcal{K}^b_N(\operatorname{Proj}(\mathcal{E})) \to \mathcal{D}_N^{\mathrm{perf}}(\mathcal{E})$ is an equivalence.
(b) The canonical triangle functor $\mathcal{D}_N^{\mathrm{perf}}(\mathcal{E}) \to \mathcal{D}^b_N(\mathcal{E})$ is fully faithful.

Proof.

(a) The claim follows from Lemma 3.10 applied to $\mathrm{Proj}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Proj}(\mathcal{E}))$.

(b) The functor exists due to Lemma 3.6. By precomposition with the equivalence from (a) the claim follows from Lemma 3.10. □

Remark 3.19. Consider a perfect N-complex $P\colon P^m \longrightarrow \cdots \longrightarrow P^n \in \mathcal{D}_N^{\mathrm{perf}}(\mathcal{E})$ over an exact idempotent complete category, then there is a distinguished triangle,

$$\mu_1^n(P^n) \longrightarrow P \longrightarrow \tau^{\leq n-1} P \longrightarrow \Sigma\mu_1^n(P^n),$$

in $\mathcal{D}_N(\mathcal{E})$. Hence, any such P is an iterated extension of the object $\mu_1^s(P^s)$, where $s \in \{m, \ldots, n\}$. In particular, as Hom is a (co)homological functor,

$$\mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(-, P) = 0 \iff \mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(-, \mu_1^s(P^s)) = 0 \text{ for all } s \in \{m, \ldots, n\},$$

$$\mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(P, -) = 0 \iff \mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(\mu_1^s(P^s), -) = 0 \text{ for all } s \in \{m, \ldots, n\}.$$

Definition 3.20. The **triangulated hull** $\mathrm{tri}_{\mathcal{T}}(\mathcal{S})$ of subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is the smallest triangulated subcategory of $\mathcal{T}$ containing all objects of $\mathcal{S}$.

Lemma 3.21. *The category of perfect complexes over an exact idempotent complete category $\mathcal{E}$ is given by* $\mathcal{D}_N^{\mathrm{perf}}(\mathcal{E}) = \mathrm{tri}\{\mu_1^s(P) \mid P \in \mathrm{Proj}(\mathcal{E}), s \in \{1, \ldots, N-1\}\}$, *see also [IKM17, Lem. 2.6.(ii)].*

Proof. By Lemma 1.60 and Remark 3.19, we have

$$\mu_1^0(P) = \Sigma\mu_{N-1}^{N-1}(P) \in \mathrm{tri}\{\mu_1^s(P) \mid P \in \mathrm{Proj}(\mathcal{E}), s \in \{1, \ldots, N-1\}\}$$

and the claim follows from Remark 3.19 using Theorem 1.59. □

Definition 3.22. We define the **stabilized *N*-derived category**, or ***N*-singularity category**, of an exact idempotent complete category $\mathcal{E}$ as the Verdier quotient $\underline{\mathcal{D}}_N^b(\mathcal{E}) := \mathcal{D}_N^b(\mathcal{E})/\mathcal{D}_N^{\mathrm{perf}}(\mathcal{E})$.

Definition 3.23. Let $\mathcal{E}$ be an exact idempotent complete category. The functor $\iota^n = \iota^n_{\mathcal{E}}$, see Notation 1.69, sends $\mathrm{Proj}(\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E}))$ to $\mathcal{D}_N^{\mathrm{perf}}(\mathcal{E})$, see Theorem 1.71.(b), and hence factors uniquely through the stable category as $\underline{\iota}^n = \underline{\iota}^n_{\mathcal{E}}$:

$$\begin{tikzcd}
\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E}) \arrow[r, "\iota^n_{\mathcal{E}}"] \arrow[d] & \mathcal{D}_N^b(\mathcal{E}) \arrow[d] \\
\underline{\mathrm{Mor}^{\mathrm{m}}}_{N-2}(\mathcal{E}) \arrow[r, dashed, "\underline{\iota}^n_{\mathcal{E}}"] & \underline{\mathcal{D}}_N^b(\mathcal{E})
\end{tikzcd}$$

Definition 3.24. For $n \in \mathbb{Z}$, we define the ***n*th extension group** of two N-complexes $X, Y \in C_N(\mathcal{E})$ over an exact idempotent complete category $\mathcal{E}$ as $\mathrm{Ext}^n_{\mathcal{E}}(X, Y) := \mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(X, \Sigma^n Y)$.

Part (a) of Lemma 3.25 yields in particular a statement of Orlov's in our setting, see [Orl09, Lem. 1.10]. It also serves to generalize Buchweitz's characterization of perfect 2-complexes of modules in part (b), see [Buc21, Lem. 1.2.1].

Lemma 3.25. *Let $\mathcal{E}$ be an exact idempotent complete category with enough projectives.*

(a) If $P \in C_N^{-,b_{\mathcal{E}}}(\mathrm{Proj}(\mathcal{E}))$ is acyclic at all positions up to $n+1 \in \mathbb{Z}$, then $P \cong \iota^n \Omega^{n+1} P$ in $\underline{\mathcal{D}}_N^b(\mathcal{E})$. In particular, any object of $\underline{\mathcal{D}}_N^b(\mathcal{E})$ lies in $\underline{\iota}^n(\underline{\mathrm{Mor}^{\mathrm{m}}}_{N-2}(\mathcal{E}))$, for any sufficiently small $n \in \mathbb{Z}$.

(b) An N-complex $X \in \mathcal{D}_N^b(\mathcal{E})$ lies in $\mathcal{D}_N^{\mathrm{perf}}(\mathcal{E})$ if and only if $\mathrm{Ext}^n_{\mathcal{E}}(X, \iota^0(\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E}))) = 0$ for any sufficiently large $n \in \mathbb{Z}$. In particular, the subcategory $\mathcal{D}_N^{\mathrm{perf}}(\mathcal{E})$ of $\mathcal{D}_N^b(\mathcal{E})$ is thick.

Proof.

(a) By Corollary 2.32 and Lemma 3.8, there is a distinguished triangle

$$\sigma^{\le n} P \longrightarrow P \longrightarrow \sigma^{\ge n-N+2} P \longrightarrow \Sigma \sigma^{\le n} P$$

in $\mathcal{D}_N^{-,b}(\mathcal{E})$ with $\sigma^{\le n} P$ acyclic. It follows that $P \cong \sigma^{\ge n-N+2} P$ in $\mathcal{D}_N(\mathcal{E})$. Setting $C := \Omega^{n+1} P$ we have $\tau^{\le n} \sigma^{\ge n-N+2} P = \iota^n C$ and $\tau^{\ge n+1} \sigma^{\ge n-N+2} P = \tau^{\ge n+1} P$. Applying Remark 3.15 to $X = \sigma^{\ge n-N+2} P \cong P$ yields a distinguished triangle

$$\tau^{\ge n+1} P \longrightarrow P \longrightarrow \iota^n C \longrightarrow \Sigma \tau^{\ge n+1} P \tag{3.2}$$

in $\mathcal{D}_N^b(\mathcal{E})$ with $\tau^{\ge n+1} P \in \mathcal{D}_N^{\mathrm{perf}}(\mathcal{E})$. It follows that $P \cong \iota^n C$ in $\underline{\mathcal{D}}_N^b(\mathcal{E})$, which proves (a). The particular claim follows by Theorem 3.11.(b).

(b) Assume first that $X \in \mathcal{D}_N^{\mathrm{perf}}(\mathcal{E})$. It follows from Theorem 1.59 and Lemma 3.10 that, for any sufficiently large $n \in \mathbb{Z}$, and $s \in \{0, 1\}$,

$$\mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(X, \Sigma^{2n+s} \iota^0(\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E}))) \cong \mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(\Sigma^{-s} X, \Theta^{Nn} \iota^0(\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E}))) = 0,$$

because these two complexes have disjoint support. To show the converse suppose that $X \in \mathcal{D}_N^b(\mathcal{E})$ satisfies the vanishing of extension groups. Corollaries 2.43 and 3.3 yield a projective resolution $P \to X$ with $P \in C_N^{-,b_{\mathcal{E}}}(\mathrm{Proj}(\mathcal{E}))$ and $X \cong P$ in $\mathcal{D}_N(\mathcal{E})$. Pick $m \in \mathbb{Z}$ sufficiently large such that P is acyclic at all positions up to $n+1$ where $n = -Nm$. Due to the distinguished triangle (3.2) from the proof of (a), it suffices to show that $\iota^n C \in \mathcal{D}_N^{\mathrm{perf}}(\mathcal{E})$. Then X lies in the triangulated subcategory $\mathcal{D}_N^{\mathrm{perf}}(\mathcal{E})$ of $\mathcal{D}_N^b(\mathcal{E})$. Since $\iota^n C \cong \Sigma^{2m} \iota^0 C \in \Sigma^{2m} \iota^0 \mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E})$ by Theorem 1.59, we may assume by hypothesis that $\mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(P, \iota^n C) = 0$. Then (3.2) yields an induced morphism, see Remark 1.85,

$$\begin{tikzcd}
P \arrow[r, "0"] \arrow[dr, "0"'] & \iota^n C \arrow[r] \arrow[d, equal] & \Sigma \tau^{\ge n+1} P \arrow[dl, dashed, "f"] \\
& \iota^n C &
\end{tikzcd}$$

in $\mathcal{D}_N(\mathcal{E})$. Lemma 3.10 shows that

$$\mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(\tau^{\ge n+1} \Sigma \tau^{\ge n+1} P, \iota^n C) \cong \mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(\tau^{\ge n+1} \Sigma \tau^{\ge n+1} P, \iota^n C) = 0,$$

because these two complexes have disjoint support. Thus, there is an induced morphism, see Remark 3.15 and Remark 1.85,

$$\begin{tikzcd}
\tau^{\geq n+1}\Sigma\tau^{\geq n+1}P \arrow[r] \arrow[dr, "0"'] & \Sigma\tau^{\geq n+1}P \arrow[r] \arrow[d, "f"] & \tau^{\leq n}\Sigma\tau^{\geq n+1}P \arrow[dl, dashed] \\
 & \iota^n C. &
\end{tikzcd}$$

in $\mathcal{D}_N(\mathcal{E})$. By Lemma 3.26.(b), f and hence $\mathrm{id}_{\iota^n C} = \iota^n \,\mathrm{id}_C$ thus factors through $\tau^{\leq n}\Sigma\tau^{\geq n+1}P \in \iota^n \operatorname{Proj}(\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E}))$, see Theorem 1.71.(b). By Proposition 3.13 this implies that id_C and hence C is zero in the stable category $\underline{\mathrm{Mor}^{\mathrm{m}}}_{N-2}(\mathcal{E})$. It follows that $C \in \operatorname{Proj}(\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E}))$ due to Remark 1.27.(b) and thus $\iota^n C \in \mathcal{D}^{\mathrm{perf}}_N(\mathcal{E})$ as claimed. The particular claim follows as $\mathrm{Ext}^n_{\mathcal{E}}(-,-)$ commutes with direct sums in the first argument. □

Lemma 3.26. *For an N-complex $X \in C_N(\mathcal{A})$ over an additive category $\mathcal{A}$, and $n \in \mathbb{Z}$ we have:*

(a) $\tau^{\geq n-N+1}\Sigma\tau^{\leq n}X = \bigoplus_{k=1}^{N-1} \mu^{n-k}_{N-k}(X^{n-k+1})$

(b) $\tau^{\leq n-1}\Sigma\tau^{\geq n}X \cong_{C_N(\mathcal{A})} \bigoplus_{k=1}^{N-1} \mu^{n-1}_{N-k}(X^{n+k-1})$

Proof. We use the explicit description of the considered N-complexes given in Construction 1.58.(a).

(a) The direct summand corresponding to the last row of the matrix of the differential of $\tau^{\geq n-N+1}\Sigma\tau^{\leq n}X$ is zero, which implies the claimed equality.

(b) The differentials of $\tau^{\leq n-1}\Sigma\tau^{\geq n}X$ read

$$d^k_{\tau^{\leq n-1}\Sigma\tau^{\geq n}X} = \left(\begin{array}{ccc} & E_{k-n+N} & \\ \hline -d_X^{\{k-n+N\}} & \cdots & -d_X \end{array}\right)$$

for $k \in \{n-N+1,\dots,n-2\}$. The desired isomorphism $f\colon \tau^{\leq n-1}\Sigma\tau^{\geq n}X \to \bigoplus_{k=1}^{N-1} \mu^{n-k}_{N-k}(X^{n-k+1})$ is given by

$$f^k = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ d_X & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ d_X^{\{k-n+N-1\}} & \cdots & d_X & 1 \end{pmatrix}$$

for $k \in \{n-N+1,\dots,n-1\}$ and zero elsewhere. □

3.3. **Stabilized syzygies.** In this subsection, we approach the question whether full faithfulness of $\iota^n\colon \mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E}) \to \mathcal{D}^b_N(\mathcal{E})$, see Proposition 3.13, persists under stabilization. We bypass this problem by restricting the functor $\underline{\iota}^n$ to $\underline{\Omega}^{n+1}(\underline{\mathrm{TAPC}}_N(\mathcal{E}))$, see Proposition 3.27. Following Buchweitz, this suffices to prove Theorem A. Avramov, Briggs, Iyengar and Letz offer an alternative argument, see [Buc21, p. 133], which depends on the existence of arbitrary products in the category of modules. Our proof, inspired by Orlov, see [Orl09, Prop. 1.11], does not require this assumption.

Proposition 3.27. *Let $\mathcal{E}$ be an exact idempotent complete category and $n \in \mathbb{Z}$. Then the functor $\underline{\iota}^n\colon \underline{\mathrm{Mor}^{\mathrm{m}}}_{N-2}(\mathcal{E}) \to \underline{\mathcal{D}}^b_N(\mathcal{E})$ becomes fully faithful when restricted to the image $\underline{\Omega}^{n+1}(\underline{\mathrm{TAPC}}_N(\mathcal{E}))$.*

Proof. We may assume that $n = 0$. Consider $X, Y \in \mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E})$, and write $X = \Omega^1 \tilde{P}$ and $Y = \Omega^1 \tilde{Q}$ where $\tilde{P}, \tilde{Q} \in \mathrm{TAPC}_N(\mathcal{E})$. By Lemma 3.16, we have isomorphisms

$$\iota^0 Y \cong \tau^{\leq 0}\tilde{Q} =: Q \in C_N^{-,b\mathcal{E}}(\mathrm{Proj}(\mathcal{E})) \qquad \text{and} \qquad \tau^{\leq m-1} Q \cong \iota^{m-1}\Omega^m Q$$

in $\mathcal{D}_N(\mathcal{E})$ for any sufficiently small $m \in \mathbb{Z}$. Then Remark 3.15 yields a distinguished triangle

$$\tau^{\geq m} Q \xrightarrow{\ v\ } \iota^0 Y \xrightarrow{\ t\ } \iota^{m-1}\Omega^m Q \longrightarrow \Sigma\tau^{\geq m} Q \tag{3.3}$$

in $\mathcal{D}_N(\mathcal{E})$ with $C(t) \cong \Sigma\tau^{\geq m} Q \in \mathcal{D}_N^{\mathrm{perf}}(\mathcal{E})$. We use this to prove that

$$\mathrm{Hom}_{\underline{\mathrm{Mor}}^{\mathrm{m}}_{N-2}(\mathcal{E})}(X, Y) \xrightarrow[\cong]{\iota^0} \mathrm{Hom}_{\underline{\mathcal{D}}^b_N(\mathcal{E})}(\iota^0 X, \iota^0 Y)$$

is an isomorphism.

Surjectivity: Consider a morphism $\iota^0 X \to \iota^0 Y$ in $\underline{\mathcal{D}}^b_N(\mathcal{E})$ given by a roof

$$s^{-1}g\colon\ \iota^0 X \xrightarrow{\ g\ } W \xleftarrow{\ s\ } \iota^0 Y$$

of morphisms g, s in $\mathcal{D}^b_N(\mathcal{E})$ with $C(s) \in \mathcal{D}_N^{\mathrm{perf}}(\mathcal{E})$. We may assume that $\Sigma^{-1}C(s)$ and $\iota^{m-1}\Omega^m Q$ have disjoint supports and hence

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(\Sigma^{-1}C(s), \iota^{m-1}\Omega^m Q) \cong \mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(\Sigma^{-1}C(s), \iota^{m-1}\Omega^m Q) = 0,\\
&\mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(\iota^0 X, \Sigma\tau^{\geq m} Q) \cong \mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(\tau^{\leq 0}\tilde{P}, \Sigma\tau^{\geq m} Q) = 0
\end{aligned}$$

by Lemmas 3.10, 3.16 and 3.28.(b). Using the distinguished triangles $T(s)$ from Construction 1.31 and (3.3) together with Proposition 3.13, we obtain the following induced morphisms in $\mathcal{D}_N(\mathcal{E})$, where $f \in \mathrm{Hom}_{\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E})}(X, Y)$, see Remark 1.85:

$$\begin{tikzcd}
& \iota^{m-1}\Omega^m Q & \\
\Sigma^{-1}C(s) \arrow[ur, "0"] \arrow[r] & \iota^0 Y \arrow[u, "t"'] \arrow[r, "s"] & W \arrow[ul, dashed, "r"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& \iota^0 X \arrow[dl, dashed, "\iota^0 f"'] \arrow[d, "r\circ g"] \arrow[dr, "0"] & \\
\iota^0 Y \arrow[r, "t"] & \iota^{m-1}\Omega^m Q \arrow[r] & \Sigma\tau^{\geq m} Q
\end{tikzcd}$$

These fit into an equivalence of roofs $s^{-1}g \to \iota^0 f$ in $\mathcal{D}_N(\mathcal{E})$:

$$\begin{tikzcd}
 & W \arrow[d, "r"] & \\
\iota^0 X \arrow[ur, "g"] \arrow[r, "r \circ g"] \arrow[dr, "\iota^0 f"'] & \iota^{m-1}\Omega^m Q & \iota^0 Y \arrow[l, "t"'] \arrow[ul, "s"'] \arrow[dl, equal] \\
 & \iota^0 Y \arrow[u, "t"] &
\end{tikzcd}$$

Injectivity: Suppose that $f \in \mathrm{Hom}_{\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E})}(X, Y)$ is a morphism with $\iota^0 f = 0$ in $\underline{\mathcal{D}}^b_N(\mathcal{E})$. We prove that f factors through an object of $\mathrm{Proj}(\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E}))$. By assumption, there is a morphism $s \colon \iota^0 Y \to W$ in $\mathcal{D}^b_N(\mathcal{E})$ with cone $C(s) \in \mathcal{D}^{\mathrm{perf}}_N(\mathcal{E})$ such that

$$\begin{tikzcd}
 & \iota^0 Y \arrow[d, "s"] \arrow[dr, equal] & \\
\iota^0 X \arrow[ur, "\iota^0 f"] \arrow[r, "0"] & W & \iota^0 Y \arrow[l, "s"]
\end{tikzcd}$$

commutes in $\mathcal{D}_N(\mathcal{E})$. As before, we may assume that $\mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(\Sigma^{-1}C(s), \iota^{m-1}\Omega^m Q) = 0$ and use the distinguished triangles $T(s)$ from Construction 1.31 and (3.3) to obtain the following induced morphisms in $\mathcal{D}_N(\mathcal{E})$:

$$\begin{tikzcd}
 & \iota^0 X \arrow[dl, dashed, "g"'] \arrow[d, "\iota^0 f"] \arrow[dr, "0"] & \\
\Sigma^{-1}C(s) \arrow[r, "u"] & \iota^0 Y \arrow[r, "s"] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \Sigma^{-1}C(s) \arrow[dl, dashed, "w"'] \arrow[d, "u"] \arrow[dr, "0"] & \\
\tau^{\geq m}Q \arrow[r, "v"] & \iota^0 Y \arrow[r, "t"] & \iota^{m-1}\Omega^m Q
\end{tikzcd}$$

These fit into a commutative diagram

$$\begin{tikzcd}
 & \tau^{\geq m}Q \arrow[dr, "v"] & \\
\iota^0 X \arrow[ur, "w \circ g"] \arrow[r, "g"] \arrow[dr, "\iota^0 f"'] & \Sigma^{-1}C(s) \arrow[u, "w"] \arrow[r, "u"] \arrow[d, "u"] & \iota^0 Y \arrow[dl, equal] \\
 & \iota^0 Y &
\end{tikzcd}$$

in $\mathcal{D}_N(\mathcal{E})$, and hence $\iota^0 f$ factors through $\tau^{\geq m}Q \in \mathcal{D}_N^{\mathrm{perf}}(\mathcal{E})$. The claimed injectivity follows from Proposition 3.13 and Lemma 3.29. □

Lemma 3.28. *Consider N-complexes* $X \in C_N^{\varnothing^*}(\mathcal{E})$, $\tilde{P} \in \mathrm{TAPC}_N(\mathcal{E})$, $Q \in C_N^b(\mathrm{Proj}(\mathcal{E}))$ *over an exact idempotent complete category* $\mathcal{E}$ *and* $n \in \mathbb{Z}$. *We have*

(a) $\mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(X, Q) = 0$ *and, in particular,* $\mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(\tau^{\leq n}\tilde{P}, \tau^{\leq n-N+1}Q) = 0$,

(b) $\mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(\tau^{\leq n}X, \Sigma\tau^{\leq n}Q) = 0$ *and, in particular,* $\mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(\tau^{\leq n}\tilde{P}, \Sigma\tau^{\leq n}Q) = 0$.

Proof.

(a) We may assume that $Q \neq 0$ and set $m := \max\{k \in \mathbb{Z} \mid Q^k \neq 0\}$. Since X is totally acyclic, $\mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(X, \mu_1^m(Q^m)) = H_{(1)}^{-m}(\mathrm{Hom}_{\mathcal{E}}(X, Q^m)) = 0$ due to Remark 1.63.(d). By induction on the length of Q we may suppose that $\mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(X, \tau^{\leq m-1}Q) = 0$. Then $\mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(X, Q) = 0$ follows by applying $\mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(X, -)$ to the distinguished triangle

$$\mu_1^m(Q^m) \longrightarrow Q \longrightarrow \tau^{\leq m-1}Q \longrightarrow \Sigma\mu_1^m(Q^m)$$

in $\mathcal{K}_N(\mathcal{E})$, see Remark 3.15. The second claim follows using Lemma 3.10:

$$\mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(\tau^{\leq n}\tilde{P}, \tau^{\leq n-N+1}Q) \cong \mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(\tau^{\leq n}\tilde{P}, \tau^{\leq n-N+1}Q) = \mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(\tilde{P}, \tau^{\leq n-N+1}Q) = 0.$$

(b) We may assume that $n = 0$ and $Q = \tau^{\leq 0}Q$. We have $\tau^{\geq -N+1}\Sigma Q = \bigoplus_{k=1}^{N-1} \mu_{N-k}^{-k}(Q^{-k+1})$ by Lemma 3.26.(a), see Construction 1.58.(a). Using Remark 1.63.(d), this implies that

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(\tau^{\leq 0}X, \tau^{\geq -N+1}\Sigma Q) &\cong \bigoplus_{k=1}^{N-1} \mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(\tau^{\leq 0}X, \mu_{N-k}^{-k}(Q^{-k+1})) \\
&\cong \bigoplus_{k=1}^{N-1} H_{(N-k)}^{k}(\mathrm{Hom}_{\mathcal{E}}(\tau^{\leq 0}X, Q^{-k+1})) \\
&= \bigoplus_{k=1}^{N-1} H_{(N-k)}^{k}(\tau^{\geq 0}\,\mathrm{Hom}_{\mathcal{E}}(X, Q^{-k+1})) \\
&= \bigoplus_{k=1}^{N-1} H_{(N-k)}^{k}(\mathrm{Hom}_{\mathcal{E}}(X, Q^{-k+1})) \\
&= 0,
\end{aligned}$$

since X is totally acyclic. Due to part (a) for $\tau^{\leq -N}\Sigma Q \in C_N^b(\mathrm{Proj}(\mathcal{E}))$, we have

$$\mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(\tau^{\leq 0}X, \tau^{\leq -N}\Sigma Q) = \mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(X, \tau^{\leq -N}\Sigma Q) = 0.$$

Then $\mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(\tau^{\leq 0}X, \Sigma Q) = 0$ by applying $\mathrm{Hom}_{\mathcal{K}_N(\mathcal{E})}(\tau^{\leq 0}X, -)$ to the distinguished triangle

$$\tau^{\geq -N+1}\Sigma Q \longrightarrow \Sigma Q \longrightarrow \tau^{\leq -N}\Sigma Q \longrightarrow \Sigma\tau^{\geq -N+1}\Sigma Q$$

in $\mathcal{K}_N(\mathcal{E})$, see Remark 3.15. The second claim is again due to Lemma 3.10. □

Lemma 3.29. *Let $\mathcal{E}$ be an exact idempotent complete category, $X = \Omega^1\tilde{P} \in \mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E})$ where $\tilde{P} \in \mathrm{TAPC}_N(\mathcal{E})$ and $Q = \tau^{\leq 0}Q \in \mathcal{D}^{\mathrm{perf}}_N(\mathcal{E})$. Then any morphism $g\colon \iota^0 X \to Q$ in $\mathcal{D}^b_N(\mathcal{E})$ factors through an object of $\iota^0\,\mathrm{Proj}(\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E}))$*

Proof. Since $\mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(\iota^0 X, \tau^{\leq -N+1}Q) \cong \mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(\tau^{\leq 0}\tilde{P}, \tau^{\leq -N+1}Q) = 0$ by Lemmas 3.16 and 3.28.(a), the morphism g factors in $\mathcal{D}_N(\mathcal{E})$ as

$$\begin{tikzcd}
 & \iota^0 X \arrow[ld, dashed] \arrow[d, "g"] \arrow[rd, "0"] & \\
\tau^{\geq -N+2}Q \arrow[r] & Q \arrow[r] & \tau^{\leq -N+1}Q,
\end{tikzcd}$$

see Remarks 1.85 and 3.15. We may thus assume that $Q = \tau^{\geq -N+2}Q$. Then applying the exact functor $\tau^{\leq 0}$ to the termsplit short exact sequence $\Sigma^{-1}Q \rightarrowtail P(Q) \twoheadrightarrow Q$, see Constructions 1.31 and 1.51, yields a distinguished triangle

$$\tau^{\leq 0}\Sigma^{-1}Q \longrightarrow \tau^{\leq 0}P(Q) \longrightarrow Q \longrightarrow \Sigma\tau^{\leq 0}\Sigma^{-1}Q$$

in $\mathcal{D}_N(\mathcal{E})$, see Lemma 1.33. Since $\mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(\iota^0 X, \Sigma\tau^{\leq 0}\Sigma^{-1}Q) = \mathrm{Hom}_{\mathcal{D}_N(\mathcal{E})}(\tau^{\leq 0}\tilde{P}, \Sigma\tau^{\leq 0}\Sigma^{-1}Q) = 0$ by Lemmas 3.16 and 3.28.(b), the morphism g factors in $\mathcal{D}_N(\mathcal{E})$ as

$$\begin{tikzcd}
 & \iota^0 X \arrow[ld, dashed] \arrow[d, "g"] \arrow[rd, "0"] & \\
\tau^{\leq 0}P(Q) \arrow[r] & Q \arrow[r] & \Sigma\tau^{\leq 0}\Sigma^{-1}Q,
\end{tikzcd}$$

see Remark 1.85, where $\tau^{\leq 0}P(Q) = \bigoplus_{k=1}^{N-1}\mu^0_k(Q^{-k+1}) \in \iota^0\,\mathrm{Proj}(\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{E}))$, see Theorem 1.71.(b). □

3.4. **Stabilized truncations.** In this subsection, we prove that the stabilized truncations $\underline{\tau}^{\leq n}$ for $n \in \mathbb{Z}$ are pairwise isomorphic fully faithful triangle functors.

Proposition 3.30. *Let $\mathcal{E}$ be an exact idempotent complete category and $n \in \mathbb{Z}$. Then the truncation $\tau^{\leq n}\colon C_N(\mathcal{E}) \to C_N(\mathcal{E})$ induces a triangulated functor $\underline{\tau}^{\leq n}\colon \underline{\mathrm{APC}}_N(\mathcal{E}) \to \underline{\mathcal{D}}^b_N(\mathcal{E})$ such that $\underline{\tau}^{\leq n} \cong \underline{\iota}^n \circ \underline{\Omega}^{n+1}$.*

Proof. Due to Lemmas 2.49 and 3.16, Definition 3.23 and Theorem 3.9, $\tau^{\leq n}$ induces a well-defined functor $\underline{\mathrm{APC}}_N(\mathcal{E}) \to \underline{\mathcal{D}}^{-,b}_N(\mathcal{E}) \simeq \underline{\mathcal{D}}^b_N(\mathcal{E})$ with $\underline{\tau}^{\leq n} \cong \underline{\iota}^n \circ \underline{\Omega}^{n+1}$. To prove that $\underline{\tau}^{\leq n}$ is triangulated we may assume that $n = 0$. Let Σ_A and Σ_D be the suspension functors in $\underline{\mathrm{APC}}_N(\mathcal{E})$ and $\underline{\mathcal{D}}^b_N(\mathcal{E})$, respectively. We need to establish a natural isomorphism $\eta\colon \underline{\tau}^{\leq 0}\Sigma_A \to \Sigma_D\underline{\tau}^{\leq 0}$ under which the functor $\underline{\tau}^{\leq 0}$ maps distinguished triangles to distinguished triangles. To simplify notation, we abbreviate $X^{\leq 0} := \tau^{\leq 0}X$

for an N-complex X and likewise for morphisms. For any $P \in \mathrm{APC}_N(\mathcal{E})$, there is an isomorphism

$$I(P)^{\leq 0} = \left(\bigoplus_{k \in \mathbb{Z}} \mu_N^k(P^k) \right)^{\leq 0} \cong \bigoplus_{k=-\infty}^{0} \mu_N^k(P^k) \oplus J(P) = I(P^{\leq 0}) \oplus J(P), \text{ where } J(P) := \bigoplus_{k=1}^{N-1} \mu_{N-k}^0(P^k).$$

Let $f\colon P \to Q$ be a morphism in $\mathrm{APC}_N(\mathcal{E})$. We apply the exact functor $\tau^{\leq 0}$ to $(D(f))$, see Constructions 1.31 and 1.51, and combine it with $(D(f^{\leq 0}))$. This yields a diagram of termsplit short exact sequences, in which the dashed and dotted morphisms remain to be constructed:

$$\begin{tikzcd}
P^{\leq 0} \arrow[rr, tail] \arrow[d, "f^{\leq 0}"'] \arrow[ddr, equal] \arrow[drr, phantom, "\square"] & & I(P^{\leq 0}) \arrow[rr, two heads] \arrow[d] & & \Sigma(P^{\leq 0}) \arrow[d, equal] & & \\
Q^{\leq 0} \arrow[rr, tail] \arrow[ddr, equal] & & C(f^{\leq 0}) \arrow[rr, two heads] & & \Sigma(P^{\leq 0}) & & \\
& P^{\leq 0} \arrow[rr, tail] \arrow[d, "f^{\leq 0}"] \arrow[drr, phantom, "\square"] & & I(P)^{\leq 0} \arrow[rr, two heads] \arrow[d] \arrow[uul, two heads, "p"'] & & (\Sigma P)^{\leq 0} \arrow[d, equal] \arrow[uul, dotted, two heads, "\eta_P"'] & \\
& Q^{\leq 0} \arrow[rr, tail] & & C(f)^{\leq 0} \arrow[rr, two heads] \arrow[uul, dashed, two heads] & & (\Sigma P)^{\leq 0} \arrow[uul, dotted, two heads, "\eta_P"'] & \\
& & & & J(P) \arrow[rr, equal] \arrow[d, equal] \arrow[uul, tail] & & J(P) \arrow[d, equal] \arrow[uul, dotted, tail] \\
& & & & J(P) \arrow[rr, equal] \arrow[uul, dashed, tail] & & J(P) \arrow[uul, dotted, tail]
\end{tikzcd} \tag{3.4}$$

The lower pushout is due to exactness of $\tau^{\leq 0}$, see Proposition 1.17. The dashed morphisms occur in the following commutative diagram of short exact sequences obtained from Proposition 1.12.(a) and Lemma 1.15:

$$\begin{tikzcd}
0 \arrow[r, tail] \arrow[d, tail] & J(P) \arrow[r, equal] \arrow[d, tail] & J(P) \arrow[d, dashed, tail] \\
P^{\leq 0} \arrow[r, tail] \arrow[d, equal] & Q^{\leq 0} \oplus I(P)^{\leq 0} \arrow[r, two heads] \arrow[d, "\begin{pmatrix} \mathrm{id} & 0 \\ 0 & p \end{pmatrix}"] & C(f)^{\leq 0} \arrow[d, dashed, two heads] \\
P^{\leq 0} \arrow[r, tail] & Q^{\leq 0} \oplus I(P^{\leq 0}) \arrow[r, two heads] & C(f^{\leq 0})
\end{tikzcd}$$

Then (3.4) commutes, save the dotted morphisms. Both dotted termsplit short exact sequences arise from Lemma 1.15 as well. They agree by precomposing with the epic $I(P)^{\leq 0} \twoheadrightarrow (\Sigma P)^{\leq 0}$.

In the singularity category $\underline{\mathcal{D}}^b_N(\mathcal{E})$, the perfect N-complex $J(P)$ is zero and (3.4) yields an isomorphism of candidate triangles:

$$\begin{tikzcd}
P^{\leq 0} \arrow[r] \arrow[d, equal] & Q^{\leq 0} \arrow[r] \arrow[d, equal] & C(f)^{\leq 0} \arrow[r] \arrow[d, "\cong"] & (\Sigma P)^{\leq 0} \arrow[d, "\eta_P"', "\cong"] \\
P^{\leq 0} \arrow[r] & Q^{\leq 0} \arrow[r] & C(f^{\leq 0}) \arrow[r] & \Sigma(P^{\leq 0})
\end{tikzcd}$$

The naturality of the isomorphisms $\eta = (\eta_P)_P$ results from the commutative diagram

$$\begin{tikzcd}
 & P^{\leq 0} \arrow[rr, tail] \arrow[dd, "f^{\leq 0}" near start] & & I(P^{\leq 0}) \arrow[rr, two heads] \arrow[dd] & & \Sigma(P^{\leq 0}) \arrow[dd, "\Sigma(f^{\leq 0})"] \\
P^{\leq 0} \arrow[ur, equal] \arrow[rr, tail, crossing over] \arrow[dd, "f^{\leq 0}"] & & I(P)^{\leq 0} \arrow[ur] \arrow[rr, two heads, crossing over] & & (\Sigma P)^{\leq 0} \arrow[ur, "\eta_P"'] & \\
 & Q^{\leq 0} \arrow[rr, tail] & & I(Q^{\leq 0}) \arrow[rr, two heads] & & \Sigma(Q^{\leq 0}) \\
Q^{\leq 0} \arrow[ur, equal] \arrow[rr, tail] & & I(Q)^{\leq 0} \arrow[ur] \arrow[rr, two heads] \arrow[from=uu, crossing over] & & (\Sigma Q)^{\leq 0} \arrow[ur, "\eta_Q"'] \arrow[from=uu, crossing over, "(\Sigma f)^{\leq 0}" near start] &
\end{tikzcd}$$

in $C_N(\mathcal{E})$, see Remark 1.52, where the rightmost square commutes by precomposing with the epic $I(P)^{\leq 0} \twoheadrightarrow \Sigma(P)^{\leq 0}$. □

Lemma 3.31. *Let $\mathcal{E}$ be an exact idempotent complete category. The canonical maps induce a directed system of isomorphic functors*

$$\cdots \xrightarrow{\cong} \underline{\tau}^{\leq n+1} \xrightarrow{\cong} \underline{\tau}^{\leq n} \xrightarrow{\cong} \cdots$$

in the category $\operatorname{Func}(\underline{\mathrm{APC}}_N(\mathcal{E}), \underline{\mathcal{D}}^b_N(\mathcal{E}))$. *In particular, the inverse limit $\underline{\tau}^{\leq} := \varprojlim_k \underline{\tau}^{\leq k}$ exists and is isomorphic to any of the functors $\underline{\tau}^{\leq n}$, where $n \in \mathbb{Z}$.*

Proof. For $P \in \mathrm{APC}_N(\mathcal{E})$, there is a distinguished triangle

$$\mu^0_{N-1}(P^1) \longrightarrow C(\tau^{\leq 1}P \to \tau^{\leq 0}P) \longrightarrow C(\mathrm{id}_{\tau^{\leq 0}P}) \longrightarrow \Sigma\mu^0_{N-1}(P^1)$$

in $\underline{\mathcal{D}}^{-,b}(\mathcal{E}) \simeq \underline{\mathcal{D}}^b(\mathcal{E})$, see Construction 1.58.(a), Lemma 1.33 and Theorem 3.9. As $C(\mathrm{id}_{\tau^{\leq 0}P})$ and $\mu^0_{N-1}(P^1)$ are both zero, so is $C(\tau^{\leq 1}P \to \tau^{\leq 0}P)$. Thus, $\underline{\tau}^{\leq 1}P \cong \underline{\tau}^{\leq 0}P$ which implies the claim. □

3.5. **Buchweitz's Theorem.** In this subsection we finally prove Theorem A. The last missing ingredient is the essential surjectivity of the stabilized truncation.

Definition 3.32. We say that an exact category $\mathcal{E}$ has **locally finite $\mathcal{F}$-dimension** for a subcategory $\mathcal{F}$ if every objects $X \in \mathcal{E}$ has a projective resolution P over $\mathcal{E}$ such that $\mathrm{syz}^g_P(X) \in \mathcal{F}$ for some $g \in \mathbb{Z}$, see Definition 1.41. In particular, $\mathcal{E}$ has enough projectives.

Assumption 3.33.

(a) $\mathcal{E}$ is an exact idempotent complete category;

(b) $\mathcal{F}$ is a Frobenius category;

(c) $\mathcal{F}$ is a fully exact, replete subcategory of $\mathcal{E}$;

(d) $\mathcal{E}$ has locally finite $\mathcal{F}$-dimension and $\mathrm{Proj}(\mathcal{F}) = \mathrm{Proj}(\mathcal{E})$.

Remark 3.34. Assume 3.33 and consider X, P and g as in Definition 3.32. Resolving $\mathrm{syz}^g_P(X) \in \mathcal{F}$ by $\mathcal{E}$-projectives in $\mathcal{F}$ yields a projective resolution Q of X over $\mathcal{E}$ with $\mathrm{syz}^n_Q(X) \in \mathcal{F}$ for $n \geq g$. By Proposition 1.43, then $\mathrm{syz}^n_P(X) \oplus \tilde{Q}^n \cong \mathrm{syz}^n_Q(X) \oplus \tilde{P}^n$ lies in $\mathcal{F}$ for some $\tilde{P}^n, \tilde{Q}^n \in \mathrm{Proj}(\mathcal{E}) \subseteq \mathcal{F}$.

Lemma 3.35. *Assuming 3.33, for any* $P \in C^{\infty,+}_N(\mathrm{Proj}(\mathcal{E}))$*, there is an acyclic N-complex* $Q \in C^{-,\varnothing\mathcal{E}}_N(\mathrm{Proj}(\mathcal{E})) \cap \mathrm{Proj}(C_N(\mathcal{E}))$ *such that* $\Omega^n(P \oplus Q) \in \mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{F})$ *for any sufficiently small* $n \in \mathbb{Z}$.

Proof. By shifting P, we may assume that P is acyclic at all non-positive positions. Suppose that $n \in \mathbb{Z}$ is sufficiently small. Then the cokernels of the admissible monics in $\Omega^n(P \oplus Q)$ occur in $\Omega^{n+1}(P \oplus Q), \dots, \Omega^{n+N-1}(P \oplus Q)$. If all their objects lie in $\mathcal{F}$, then $\Omega^n(P \oplus Q) \in \mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{F})$, since $\mathcal{F}$ is fully exact in $\mathcal{E}$.

Write $n = k - Nl$, where $k \in \{-N+1, \dots, 0\}$ and $l \in \mathbb{N}$. For any $r \in \{1, \dots, N-1\}$, the projective resolution $\Theta^k \tau^{\leq k} \gamma^k_r(P)$ realizes $\Omega^n(P)^r = C^{k-Nl-N+r}_{(r)}(P)$ as a $2l$th syzygy of $C^k_{(N-r)}(P)$:

$$\begin{tikzcd}
\cdots \arrow[r] & P^{n-N+r} \arrow[rr, "d_P^{\{N-r\}}"] \arrow[dr, two heads] & & P^n \arrow[r, "d_P^{\{r\}}"] & \cdots \arrow[r, "d_P^{\{N-r\}}"] & P^{k-N} \arrow[r, "d_P^{\{r\}}"] & P^{k-N+r} \arrow[r, "d_P^{\{N-r\}}"] & P^k \arrow[rr, "d_P^{\{r\}}"] \arrow[dr, two heads] & & P^{k+r} \arrow[r] & \cdots \\
& & \Omega^n(P)^r \arrow[ur, tail] & & & & & & C^k_{(N-r)}(P) \arrow[ur, tail] & &
\end{tikzcd}$$

Due to Remark 3.34, there are $Q^n_r \in \mathrm{Proj}(\mathcal{E})$, for $r \in \{1, \dots, N-1\}$, such that $\Omega^n(P)^r \oplus Q^n_r \in \mathcal{F}$. As $\mathrm{Proj}(\mathcal{E}) \subseteq \mathcal{F}$, the object $Q^n := \bigoplus_{r=1}^{N-1} Q^n_r$ works for all r. Due to Remark 2.48, we have $\Omega^n(\mu^n_N(Q^n)) = \mu_{N-1}(Q^n)$ and $\Omega^n(P \oplus \mu^n_N(Q^n))^r = \Omega^n(P)^r \oplus Q^n \in \mathcal{F}$ for any r. The claim follows with $Q := \bigoplus_{n \ll 0} \mu^n_N(Q^n)$, see Remarks 1.21 and 2.10.(b) and Lemma 1.50. □

Proposition 3.36. *Assuming 3.33, the restricted truncation* $\underline{\tau}^{\leq} : \underline{\mathrm{APC}}_N(\mathcal{F}) \to \underline{\mathcal{D}}^b_N(\mathcal{E})$ *is essentially surjective, see Corollary 2.22.*

Proof. Due to Corollaries 2.43 and 3.3, any $X \in C^b_N(\mathcal{E})$ admits a projective resolution $P \to X$ with $P \in C^{-,b\mathcal{E}}_N(\mathrm{Proj}(\mathcal{E}))$. By Lemma 3.35, we may assume that $\Omega^{n+1}P \in \mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{F})$ for any sufficiently small n. Proposition 2.47 yields a $Q \in \mathrm{APC}_N(\mathcal{F})$ with $\Omega^{n+1}Q \cong \Omega^{n+1}P$ in $\mathrm{Mor}^{\mathrm{m}}_{N-2}(\mathcal{F})$. Using Lemmas 3.25.(a), 3.16 and 3.31, we conclude that $X \cong P \cong \underline{\iota}^n \underline{\Omega}^{n+1} Q \cong \underline{\tau}^{\leq n} Q \cong \underline{\tau}^{\leq} Q$ in $\underline{\mathcal{D}}^b_N(\mathcal{E})$ for sufficiently small $n \in \mathbb{Z}$. □

Theorem 3.37 (Buchweitz's Theorem)**.** *Assuming 3.33, there is, for any* $n \in \mathbb{Z}$*, the following commutative diagram, where* $\simeq$ *indicates triangle equivalences:*

$$\begin{tikzcd}
\underline{\mathrm{APC}}_N(\mathcal{F}) \arrow[rr, "\underline{\Omega}^{n+1}_{\mathcal{F}}", "\simeq"'] \arrow[dd, hook] \arrow[dr, "\simeq"] & & \underline{\mathrm{Mor}^{\mathrm{m}}}_{N-2}(\mathcal{F}) \arrow[dl, "\simeq"'] \arrow[dd, hook] \\
& \underline{\mathcal{D}}^b_N(\mathcal{E}) & \\
\underline{\mathrm{TAPC}}_N(\mathcal{E}) \arrow[rr, "\underline{\Omega}^{n+1}_{\mathcal{E}}"] \arrow[ur, "\underline{\tau}^{\le n}"] & & \underline{\mathrm{Mor}^{\mathrm{m}}}_{N-2}(\mathcal{E}) \arrow[ul, "\underline{\iota}^n_{\mathcal{E}}"']
\end{tikzcd}$$

Proof. By Lemma 1.74 and Corollary 2.22, there is a fully faithful functor $\underline{\mathrm{Mor}^{\mathrm{m}}}_{N-2}(\mathcal{F}) \to \underline{\mathrm{Mor}^{\mathrm{m}}}_{N-2}(\mathcal{E})$ and a fully faithful triangle functor $\underline{\mathrm{APC}}_N(\mathcal{F}) = \underline{\mathrm{TAPC}}_N(\mathcal{F}) \to \underline{\mathrm{TAPC}}_N(\mathcal{E})$, see Remark 2.3. The outer square commutes by construction. By Proposition 3.30, $\underline{\tau}^{\le n} \cong \underline{\iota}^n_{\mathcal{E}} \circ \underline{\Omega}^{n+1}_{\mathcal{E}}$ is a triangle functor. The restricted truncation $\underline{\tau}^{\le n} \colon \underline{\mathrm{APC}}_N(\mathcal{F}) \to \underline{\mathcal{D}}^b_N(\mathcal{E})$ is then a triangle functor isomorphic to $\underline{\iota}^n_{\mathcal{E}} \circ \underline{\Omega}^{n+1}_{\mathcal{F}}$. Since $\underline{\Omega}^{n+1}_{\mathcal{F}}$ is a triangle equivalence by Theorem 2.50, it is fully faithful by Proposition 3.27 and hence a triangle equivalence by Lemma 3.31 and Proposition 3.36. The restriction $\underline{\mathrm{Mor}^{\mathrm{m}}}_{N-2}(\mathcal{F}) \to \underline{\mathcal{D}}^b_N(\mathcal{E})$ of $\underline{\iota}^n_{\mathcal{E}}$ is then a triangle equivalence as well. This concludes the proof. □

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