On the symmetry of b-functions
of linear free divisors

By

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Abstract

We introduce the concept of a prehomogeneous determinant as a possibly non-reduced version of a linear free divisor. Both are special cases of prehomogeneous vector spaces. We show that the roots of the b-function are symmetric about -1 for reductive prehomogeneous determinants and for regular special linear free divisors. For general prehomogeneous determinants, we describe conditions under which this symmetry persists.

Combined with Kashiwara's theorem on the roots of b-functions, our symmetry result shows that -1 is the only integer root of the b-function. This gives a positive answer to a problem posed by Castro-Jiménez and Ucha-Enríquez in the above cases.

We study the condition of strong Euler homogeneity in terms of the action of the stabilizers on the normal spaces.

As an application of our results, we show that the logarithmic comparison theorem holds for reductive linear Koszul free divisors exactly if they are strongly Euler homogeneous.

Contents

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§1.   Logarithmic comparison theorem and b-functions

Let D be a reduced hypersurface in a complex n-dimensional manifold X, let $U = X \smallsetminus D$, and let $j : U \hookrightarrow X$ be inclusion. Then the de Rham morphism

$$\Omega_X^\bullet(*D) \to \mathbf{R}j_*\mathbb{C}_U$$

is a quasi-isomorphism. This so-called *Grothendieck's comparison theorem* is the core of the proof of Grothendieck's algebraic de Rham theorem [Gro66] in which D plays the role of a divisor at infinity for a compactification. In particular, for Stein X, each cohomology class $c \in H^k(U; \mathbb{C})$ is represented as $c(\sigma) = \int_\sigma \omega$ by a differential k-form ω with finite pole order along D. The natural question of limiting this pole order dates back to Griffiths [Gri69] and has been studied later by Deligne and Dimca [DD90, Dim91], Karpishpan [Kar91], and others.

By analogy with Grothendieck's comparison theorem, one says that the *logarithmic comparison theorem (LCT)* holds for D if the inclusion

$$\Omega_X^\bullet(\log D) \hookrightarrow \Omega_X^\bullet(*D)$$

is a quasi-isomorphism. The explicit characterization of hypersurfaces for which the LCT holds is an open problem (see [Tor07] for an overview). Mainly the extremal cases of isolated singularities and free divisors (as defined in [Sai80]) have been studied. For isolated singularities the LCT can be explicitly characterized in the quasihomogeneous case (see [HM98]) and there are partial results for the nonquasihomogeneous case (see [Sch07]). Also several variants of the LCT property have been introduced and studied. For instance, F.J. Calderón Moreno and L. Narváez Macarro [CMNM09] generalized the LCT to integrable logarithmic connections. More famous is a global algebraic version of the LCT which is conjectured to hold for all hyperplane arrangements in characteristic 0 (see [Ter78, Conj. 3.1] and [WY97, §1]).

In this article we shall be concerned with the case of free divisors. The main conjecture in this case is the following from [CMMNMCJ02, Conj. 1.4].

Conjecture 1.1 Calderón-Moreno et al.. *For a free divisor, the LCT implies strong Euler homogeneity.*

By definition, *strong Euler homogeneity at p* for some $p \in D$ means that $f \in \mathfrak{m}_{X,p} \cdot J(f)$, for some, and hence any, local defining equation $f \in \mathscr{O}_{X,p}$ at p, where $J(f)$ denotes the ideal of partials of f. By *strong Euler homogeneity of D* without reference to the point we mean strong Euler homogeneity at p for all $p \in D$ (see Definition 6.1 for more details). We have proved Conjecture for Koszul free divisors (see Section 7 for their definition) and for $n \leq 3$ in [GS06, Cor. 1.5, Thm. 1.6]. For plane curves, F.J Calderón-Moreno et al. [CMMNMCJ02, Thm. 3.5] showed that the implication in Conjecture is even an equivalence, but for $n \geq 3$ a counter-example for the converse implication is missing. In fact the first result in this direction was a stronger sufficient condition for LCT, proved by F.J. Castro-Jiménez et al. [CJNMM96], which appeared to be too strong to be necessary: Any locally quasihomogeneous free divisor satisfies the LCT. By definition, *locally quasihomogeneous* hypersurfaces admit at any point a defining equation which is quasihomogeneous with respect to strictly positive weights on a local coordinate system. As observed by L. Narváez Macarro [NM08, Rem. 1.7.4], the above implication still holds true even without this strictness requirement. The resulting condition is referred to as local weak quasihomogeneity. His argument is based on the following deep (but equally inexplicit) characterization of the LCT in terms of $\mathscr{D}_X(\log D)$-modules given in [CMNM05, Cor. 4.3].

Theorem 1.1 Calderón Moreno and Narváez Macarro. *A free divisor D satisfies the logarithmic comparison theorem if*

(a) $\mathscr{D}_X \overset{L}{\otimes}_{\mathscr{D}_X(\log D)} \mathscr{O}_X(D)$ *is concentrated in degree* 0 *and*

(b) $\mathscr{D}_X \otimes_{\mathscr{D}_X(\log D)} \mathscr{O}_X(D) \to \mathscr{O}_X(*D)$ *is injective.*

Condition 1.1.(a) is fulfilled for Koszul free D and condition 1.1.(b) is equivalent to $\mathrm{Ann}_{\mathscr{D}_X}(f^{-1})$ being generated by order one operators. This latter condition is studied in [Tor04] where the following is proved in [Tor04, Cor. 1.8] for the Koszul free case.

Theorem 1.2 Torrelli. *Let* $f \in \mathscr{O}_X$ *be a Koszul free germ on* $X = (\mathbb{C}^n, 0)$. *Then* $\mathrm{Ann}_{\mathscr{D}_X}(f^{-1})$ *is generated by order one operators if and only if*

(a) f *is Euler homogeneous,*

(b) -1 *is the only integer root of the (local) b-function of f, and*

(c) f and the symbols of a basis of $\mathrm{Der}(-\log f)$ form a $\mathrm{gr}_F \mathscr{D}_X$-regular sequence.

A promising class of examples to make the preceding theorems more explicit in order to better understand the LCT problem is that of *linear free divisors*. These are free divisors D in a complex vector space $X = V \cong \mathbb{C}^n$ such that $\mathrm{Der}(-\log D)$ admits a basis of vector fields with linear coefficients. Linear free divisors have been introduced in [BM06] where they occur as discriminants in certain quiver representation spaces. Later they have been studied in [GMNRS09] and [dGMS09]. We have proved a global version of the LCT [GMNRS09, Thm. 1.6] for reductive linear free divisors and it is conceivable that LCT holds for any linear free divisor.

The interest in linear free divisors arises from the fact that they are special cases of prehomogeneous vector spaces, which endows them with a rich additional structure that has been studied intensively. A particular feature in view of Theorem 1.2 is that their b-functions are much more explicit and satisfy, in certain cases, a functional equation due to M. Sato's fundamental theorem of prehomogeneous vector spaces. The present paper is motivated by the wish to understand b-functions of linear free divisors in order to apply Theorems 1.1 and 1.2. It is organized as follows:

In Section 2, we introduce *prehomogeneous determinants* as a slight generalization of linear free divisors for which all our arguments work without additional effort. We further study reductivity and a condition that is referred to as *special* in [dGMS09, Def. 2.1]. In Theorem 2.1, we give a more elementary proof of the implication "reductive implies special" from [dGMS09, Cor. 2.9] for the more general case of prehomogeneous determinants. In Section 3, we study b-functions of prehomogeneous determinants and prove our main result in Theorem 3.2 which states that the roots of the b-function of any reductive prehomogeneous determinant are symmetric about -1 which implies, in particular, that condition b of Theorem 1.2 holds true.

Theorem 1.3.   *The roots of the b-function of any reductive linear free divisor are symmetric about -1.*

In Section 4, we compute the b-function for all linear free divisors in dimension up to 4, and, using the mirolocal calculus from [SKKŌ81], for some prehomogeneous determinants arising from quiver theory. Based on this list of examples we expect that the symmetry property holds true for any linear free divisor.

Conjecture 1.2.   *The statement of Theorem 1.3 holds true also for*

non-reductive linear free divisors.

Sections 5 and 6 contain additional material independent of our application to the LCT: We show in Theorem 5.2 that the symmetry statement in our main result holds true also for regular special linear free divisors, where "regular" refers to the corresponding notion for prehomogeneous vector spaces (see [Kim03, Def. 2.14]). In Proposition 6.1, we give a sufficient condition for Euler homogeneity at a point in terms of the corresponding normal representation. In Section 7, we discuss condition c in Theorem 1.2 in order to finally prove the following statement on the LCT for linear free divisors.

Theorem 1.4. *A reductive linear Koszul free divisor satisfies the logarithmic comparison theorem if and only if it is strongly Euler homogeneous.*

§2. Linear free divisors and prehomogeneous determinants

A reduced hypersurface D in an n-dimensional complex vector space V is called a *linear free divisor* if the module $\mathrm{Der}(-\log D)$ of *logarithmic vector fields* has a basis of global degree 0 vector fields

$$\delta_k = A_k x = x^t A_k^t \partial_x \in \Gamma(V, \mathrm{Der}(-\log D))_0, \quad A_k \in \mathbb{C}^{n \times n}, \quad k = 1, \ldots, n.$$

Then, by Saito's criterion [Sai80, Thm. 1.8.ii],

$$f = \det(\delta_1, \ldots, \delta_n) \in \mathrm{Sym}(V^*),$$

is a homogeneous defining equation for D of degree n. Therefore we can assume that $\delta_1 = \epsilon$ is the *Euler vector field*

$$\epsilon = \frac{1}{n} \sum_{i=1}^{n} x_i \partial_{x_i}$$

and then also that $\delta_2, \ldots, \delta_n \in \Gamma(V, \mathrm{Der}(-\log f))_0$ where $\mathrm{Der}(-\log f)$ is the module of vector fields tangent to (the level sets of) f.

We denote by $G_D \subseteq \mathrm{GL}_V$ the unit component of the stabilizer of $f \in \mathbb{P}\,\mathrm{Sym}(V^*)$, and by $A_D \subseteq G_D$ that of $f \in \mathrm{Sym}(V^*)$. The following was shown in [GMNRS09, Lems. 2.2–2.4]: G_D is a linear algebraic group with Lie algebra of infinitesimal generators $\mathfrak{g}_D = \Gamma(V, \mathrm{Der}(-\log D))_0$, and the same hold for A_D with $\mathfrak{a}_D = \Gamma(V, \mathrm{Der}(-\log f))_0$. By Saito's criterion, $V \backslash D$ is a Zariski open orbit with finite isotropy groups and G_D an n-dimensional group. Conversely, any reduced discriminant of an n-dimensional linear algebraic group in GL_V with an open orbit is a linear free divisor. Thus, linear free divisors are special cases of discriminants of prehomogeneous vector spaces (G, ρ, V) characterized by the following conditions:

1. $G \subseteq \mathrm{GL}_V$ is connected with $\dim G = \dim V$,

2. $\rho = \mathrm{id} : G \hookrightarrow \mathrm{GL}_V$,

3. $f(x) = \det(d\rho(A_1)x, \ldots, d\rho(A_n)x) \in \mathrm{Sym}(V^*)$, $\mathfrak{g} = \langle A_1, \ldots, A_n \rangle$, is reduced.

We shall consider the more general objects defined by dropping the third condition. However, we shall frequently impose the additional condition of reductivity on G when we study b-functions in Section 3.

Definition 2.1.   Let $G \subseteq \mathrm{GL}_V$ be a connected algebraic group with $\dim G = \dim V = n$ such that (G, ρ, V) is a prehomogeneous vector space where $\rho = \mathrm{id} : G \hookrightarrow \mathrm{GL}_V$. Consider the relative invariant

$$f(x) = \det(d\rho(A_1)x, \ldots, d\rho(A_n)x) \in \mathrm{Sym}(V^*) \tag{2.1}$$

defined by generators $A_1, \ldots, A_n$ of the Lie algebra $\mathfrak{g}$ of G. We call the (possibly non-reduced) discriminant divisor $D \subseteq V$ defined by f a *prehomogeneous (discriminant) determinant*, and a *linear free divisor* in case f is reduced. We call D *reductive* if G is a reductive group. We set $G_D = G$, $\mathfrak{g}_D = \mathfrak{g}$, and $\rho_D = \rho$, and we denote the character of f by $\chi_D \colon G_D \to \mathbb{C}^*$ and its derivative by $d\chi_D \colon \mathfrak{g}_D \to \mathbb{C}$. We call

$$A_D = (\ker \chi_D)^\circ, \quad \mathfrak{a}_D = \ker d\chi_D, \tag{2.2}$$

the *annihilator* of D where $^\circ$ denotes the unit component.

While G_D is determined by D if D is a linear free divisor, it is part of the data of a general prehomogeneous determinant D. Note that, in Definition 2.1, f is indeed a relative invariant by [Kim03, Thm. 2.9]. Then χ_D is not trivial by [Kim03, Prop. 2.4.(2)] and we have

$$G_D/A_D \cong \mathbb{C}^*, \quad \mathfrak{g}_D/\mathfrak{a}_D \cong \mathbb{C}. \tag{2.3}$$

In particular, we may pick $A_1 \in \mathfrak{g}_D$ such that

$$d\chi_D(A_1) = 1. \tag{2.4}$$

We choose $A_1 = E/n$ if D is a linear free divisor. For $A \in \mathfrak{g}_D$, we abbreviate

$$\delta_A = \langle d\rho_D(A)x, \partial_x \rangle = x^t A^t \partial_x, \quad \delta_k = \delta_{A_k}. \tag{2.5}$$

By definition of a relative invariant [Kim03, §2.2] and [Kim03, Lem. 2.15],

$$f \circ \rho_D(g) = \chi_D(g) \cdot f, \quad \delta_A(f) = \langle d\rho_D(A)x, \mathrm{grad}_x(f) \rangle = d\chi_D(A) \cdot f. \quad (2.6)$$

As $\deg f = n$ by (2.1) and $G_D \subseteq \mathrm{GL}_V$ with $\dim V = n$ by Definition 2.1, we have

$$\deg \chi_D = n = \deg(\det \circ \rho_D). \quad (2.7)$$

Lemma 2.1. *For a prehomogeneous determinant D, $\chi_D = \det \circ \rho_D$ if and only if $\mathrm{tr} \circ d\rho_D = 0$ on $\mathfrak{a}_D$.*

Proof. The converse being trivial, let us assume that the second condition holds. As $d \det = \mathrm{tr}$, this gives using (2.2) that

$$\ker d(\det \circ \rho_D) \supseteq \mathfrak{a}_D = \ker d\chi_D, \quad \ker(\det \circ \rho_D) \supseteq A_D = \ker \chi_D.$$

Then both χ_D and $\det \circ \rho_D$ induce characters on G_D/A_D. But the latter has character group isomorphic to $\mathbb{Z}$ by (2.3). Using (2.7), this implies the equality $\chi_D = \det \circ \rho_D$. $\qquad\square$

Lemma 2.2. *If D is a linear free divisor or a reductive prehomogeneous determinant, one can choose $A_1 \in \mathfrak{g}_D$ with (2.4) such that*

$$\mathfrak{g}_D = \mathbb{C} \cdot A_1 \oplus \mathfrak{a}_D, \quad \mathfrak{a}_D = \langle A_2, \ldots, A_n \rangle. \quad (2.8)$$

Proof. Choosing $A_1 = E/n$, the first statement follows from (2.1) and Saito's criterion.

By [Jac79, Ch. III, §7, Thm. 10], reductivity of $\mathfrak{g}_D$ means that

$$\mathfrak{g}_D = \mathfrak{c}_D \oplus \mathfrak{s}_D \quad (2.9)$$

where $\mathfrak{c}_D$ is the center of $\mathfrak{g}_D$, $d\rho_D(\mathfrak{c}_D)$ consists of semisimple endomorphisms, and $\mathfrak{s}_D$ is a semisimple ideal. By semisimplicity of $\mathfrak{s}_D$, we have

$$\mathfrak{s}_D = [\mathfrak{s}_D, \mathfrak{s}_D]. \quad (2.10)$$

Then it follows from (2.2) that

$$\mathfrak{s}_D = [\mathfrak{g}_D, \mathfrak{g}_D] \subseteq \mathfrak{a}_D \quad (2.11)$$

and hence $A_1 \notin \mathfrak{s}_D$ by (2.4). Using (2.9) and (2.11), this shows that $A_1 \in \mathfrak{c}_D$ after subtracting a suitable element of $\mathfrak{s}_D$. $\qquad\square$

In order to say more, we shall focus on the reductive case for the remainder of this section. The following result is proved in [dGMS09, Cor. 2.9] in the case of linear free divisors.

Theorem 2.1.   *Let D be a reductive prehomogeneous determinant. Then*

$$\chi_D = \det \circ \rho_D, \quad d\chi_D = \operatorname{tr} \circ d\rho_D \qquad (2.12)$$

which means that $A_D \subseteq \mathrm{SL}_V$.

Proof.   We resume the notation $\mathfrak{c}_D$ and $\mathfrak{s}_D$ from Lemma 2.2. By (2.11), $\mathfrak{s}_D \subseteq \mathfrak{a}_D$ and hence $\mathfrak{a}_D = \mathfrak{c}'_D \oplus \mathfrak{s}_D$ where $\mathfrak{c}'_D$ is the center of $\mathfrak{a}_D$.

Now let $\mathfrak{h}_D$ be a Cartan subalgebra of $\mathfrak{s}_D$. Recall that $A_1 \in \mathfrak{c}_D$ by Lemma 2.2. Pick generators $S_1 = A_1, S_2, \ldots, S_s$ of $\mathfrak{c}_D + \mathfrak{h}_D$ such that $d\rho_D(S_1), \ldots, d\rho_D(S_s)$ are semisimple and extend them to a basis $N_1, \ldots, N_r$, $r = n - s$, of $\mathfrak{a}_D$ such that $N_1, \ldots, N_r$ are $(\mathfrak{c}_D + \mathfrak{h}_D)$-homogeneous and $d\rho_D(N_1), \ldots, d\rho_D(N_r)$ are nilpotent. Then, by (the proof of) [GMNRS09, Thm. 6.1.(4)], we have

$$d\chi_D(S_i) = \operatorname{tr} \circ d\rho_D(S_i) + \sum_{j=1}^{r} [S_i, N_j]/N_j, \quad i = 1, \ldots, s. \qquad (2.13)$$

This implies that $d\chi_D = \operatorname{tr} \circ d\rho_D$ on $\mathfrak{c}_D$. But on $\mathfrak{s}_D$, both $d\chi_D$ and $\operatorname{tr} \circ d\rho_D$ are zero by (2.11) and (2.10). Now the claim follows from Lemma 2.1.   □

Our motivation to study the symmetry of b-functions of prehomogeneous determinants stems from M. Sato's fundamental theorem of prehomogeneous vector spaces [Kim03, Prop. 4.18]. We shall need the following result to show that it holds for reductive prehomogeneous determinants in Theorem 3.1.

Theorem 2.2.   *Let D be a reductive linear free divisor.*

(a) *The Lie algebra representation ρ_D is defined over $\mathbb{Q}$ with respect to some $V_{\mathbb{Q}}$.*

(b) *With respect to any $V_{\mathbb{Q}}$ as in (a), also D is defined over $\mathbb{Q}$.*

(c) *With respect to any $V_{\mathbb{Q}}$ as in (a), also G_D is defined over $\mathbb{Q}$.*

For the convenience of the reader we give a proof of the following general fact which is probably well-known to specialists.

Lemma 2.3.   *Every finite representation V of a complex semisimple Lie algebra $\mathfrak{s}$ is defined over $\mathbb{Q}$ with respect to some $V_{\mathbb{Q}}$.*

Proof.   Let $\mathfrak{h}$ be a Cartan and $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{s}$. By Chevalley's Normalization [Ser01, Ch. VI, §6, Thm. 11] $\mathfrak{s}$ has a rational form $\mathfrak{s}_{\mathbb{Q}}$. By

complete reducibility we may assume that V is an irreducible $\mathfrak{s}$-module. Let $V_\omega = U(\mathfrak{s}) \otimes_\mathfrak{b} L_\omega$ where $U(\mathfrak{s})$ denotes the universal enveloping algebra of $\mathfrak{s}$ and L_ω is a complex one dimensional $\mathfrak{b}$-module of weight $\omega \in \mathfrak{h}^*$. Then set $E_\omega = V_\omega/N_\omega$ where N_ω is generated by all $\mathfrak{s}$-modules strictly contained in V_ω. By [Ser01, Ch. VII, §3, Thm. 2], $V = E_\omega$ where ω is integer on the coroots in $\mathfrak{h}_\mathbb{Q}$ by [Ser01, Ch. VII, §4, Prop. 3.(b)]. This implies that V_ω is a direct sum of rational weight spaces for $\mathfrak{h}_\mathbb{Q}$ and is thus defined over $\mathbb{Q}$. Then N_ω is the maximal $\mathfrak{s}$-module in V_ω that meets L_ω in zero. Choosing a $\mathbb{Q}$-basis of L_ω, V_ω also decomposes as a direct sum of $\mathbb{Q}$-vector spaces $U(\mathfrak{s}_\mathbb{Q}) \otimes_{\mathfrak{b}_\mathbb{Q}} L_{\omega,\mathbb{Q}}$ compatible with the above direct sum decomposition. Clearly N_ω is homogeneous with respect to both direct sums and hence defined over $\mathbb{Q}$. $\qquad\square$

Proof of Theorem 2.2. We show that ρ_D is defined over $\mathbb{Q}$ when restricted to $\mathfrak{a}_D$, which implies (a) by (2.8) and $A_1 = E/n$. Then (b) follows by (2.1), and (c) as in the proof of [GMNRS09, Lem. 2.2].

We resume the notation $\mathfrak{c}'_D$, $\mathfrak{s}_D$, and $\mathfrak{h}_D$ from the proof of Theorem 2.1. As $\mathfrak{c}'_D \oplus \mathfrak{h}_D$ is commutative, its representation on V can be diagonalized. By Lemma 2.3, the representation of $\mathfrak{s}_D$ on each $\mathfrak{c}'_D$-weight space of V is defined over $\mathbb{Q}$. So there is a $\mathbb{Q}$-subspace $V_\mathbb{Q}$ of V with respect to which $\rho_D|_{\mathfrak{s}_D}$ is defined over $\mathbb{Q}$ and with respect to which $\rho_D(\mathfrak{c}'_D \oplus \mathfrak{h}_D)$ consists of semisimple endomorphisms (with possibly complex weights). As $\mathbb{C} \cdot A_1 \oplus \mathfrak{c}'_D \oplus \mathfrak{h}_D$ is a Cartan subalgebra of $\mathfrak{g}_D$, it follows from [Ser01, Ch. III, §5, Thm. 3.(b)] that it equals its centralizer. Then [Sai71, Lem. 1.4] shows that ρ_D is defined over $\mathbb{Q}$ with respect to $V_\mathbb{Q}$ on $\mathbb{C} \cdot A_1 \oplus \mathfrak{c}'_D \oplus \mathfrak{h}_D$. The map $A \mapsto A - d\chi_D(A) \cdot A_1$ from $\mathbb{C} \cdot A_1 \oplus \mathfrak{c}'_D \oplus \mathfrak{h}_D$ to $(\mathbb{C} \cdot A_1 \oplus \mathfrak{c}'_D \oplus \mathfrak{h}_D) \cap \mathfrak{a}_D = \mathfrak{c}'_D \oplus \mathfrak{h}_D$ preserves the property of having eigenvalues in $\mathbb{Q}$. Thus, ρ_D is defined over $\mathbb{Q}$ with respect to $V_\mathbb{Q}$ also on $\mathfrak{c}'_D \oplus \mathfrak{h}_D$ and hence on $\mathfrak{a}_D$. Remark that once ρ_D is defined over $\mathbb{Q}$ on $\mathfrak{g}_D$, this also holds on the center $\mathbb{C} \cdot A_1 \oplus \mathfrak{c}'_D$ of $\mathfrak{g}_D$. $\qquad\square$

§3. b-functions of reductive prehomogeneous determinants

The (global) b-function of $f \in \mathbb{C}[x]$ is the minimal monic polynomial $B_f(s) \in \mathbb{C}[s]$ in a functional equation

$$B_f(s) \cdot f^s \in D_V[s] \cdot f^{s+1} \tag{3.1}$$

where D_V denotes the ring of algebraic differential operators on V. In general, the existence of the b-function has been established by I.N. Bernstein [Ber72] in the global and by J.E. Björk [Bjö79] in the local case. M. Kashiwara [Kas77] proved that the roots of the b-function are negative rational numbers.

Let (G, ρ, V) be a reductive prehomogeneous vector space. Then there is a compact Zariski dense subgroup $H \subseteq G$ and one can assume that $H \subseteq U(n)$ for some basis x of V. We refer to this basis x and the dual basis y of V^* as unitary coordinates. For a relative invariant f of (G, ρ, V) with character χ_f, the function $f^* \in \mathrm{Sym}(V)$ defined by

$$f^*(y) = \overline{f(\overline{y})} \tag{3.2}$$

is then a relative invariant of (G, ρ^*, V^*) with character

$$\chi_{f^*} = \chi_f^{-1} \tag{3.3}$$

by the unitarian trick (see [Kim03, §2.3]). In [Kim03, Prop. 2.22 and 2.23], this is used to show that f^* defines functional equations

$$f^*(\partial_x) \cdot f^{s+1}(x) = b_f(s) \cdot f^s(x), \tag{3.4}$$
$$f(\partial_y) \cdot (f^*)^{s+1}(y) = b_{f^*}(s) \cdot (f^*)^s(y),$$
$$b_f(s) = b_{f^*}(s), \tag{3.5}$$

as in (3.1) and that

$$\deg b_f(s) = \deg f = n. \tag{3.6}$$

From this one deduces that (G, ρ, V) is a regular prehomogeneous vector space and that any relative invariant f defining the discriminant is non-degenerate [Kim03, Prop. 2.24]. In particular, (G, ρ^*, V^*) is again a reductive and hence a regular prehomogeneous vector space. It is shown in [Gyo91, Cor. 2.5.10] that

$$b_f(s) = B_f(s), \quad b_{f^*}(s) = B_{f^*}(s). \tag{3.7}$$

Now let D be a (not necessarily reductive) prehomogeneous determinant defined by f as in Definition 2.1. Then we abbreviate

$$B_D(s) = B_f(s), \quad b_D(s) = b_f(s),$$

where the latter is defined in the presence of the functional equations (3.4).

Definition 3.1.   Let D be a prehomogeneous determinant. Consider $G_{D^*} = \rho_D^*(G_D) \subseteq \mathrm{GL}_{V^*}$ and $\rho_{D^*} = \mathrm{id} \colon G_{D^*} \hookrightarrow \mathrm{GL}_{V^*}$. Define D^* and f^* as D and f in Definition 2.1 with $(G, \rho, V) = (G_{D^*}, \rho_{D^*}, V^*)$. We call D^* the *dual determinant* of D. Note that D^* is a prehomogeneous determinant exactly if $f^* \neq 0$ or $D^* \neq V^*$.

For $A \in \mathfrak{g}_D$, we abbreviate

$$\delta_A^* = \langle d\rho_D^*(A)y, \partial_y \rangle = -y^t A \partial_y, \quad \delta_k = \delta_{A_k}, \tag{3.8}$$

as in (2.5) using (2.4) and (2.8). Note that (2.6) also holds for f, D, δ_A replaced by f^*, D^*, δ_A^*.

The following statement is [dGMS09, Prop. 3.7] in case of linear free divisors.

Proposition 3.1. *With D also D^* is a reductive prehomogeneous determinant (or a reductive linear free divisor) defined by f^* in (3.2).*

Proof. In unitary coordinates, we have $\rho_D^*(g) = (\rho_D(g)^{-1})^t = \overline{\rho_D(g)}$ for $g \in H$ and hence $d\rho_D^*(A) = -d\rho_D(A)^t = \overline{d\rho_D(A)}$ for $A \in \mathfrak{h}$ where $\mathfrak{h}$ denotes the Lie algebra of H. As $\mathfrak{h}$ spans $\mathfrak{g}$ over $\mathbb{C}$, it follows that f^* in Definition 3.1 coincides with f^* in (3.2). This shows that $f^* \neq 0$ and that f^* is reduced if and only if f is reduced. $\qquad\square$

If D is a reductive prehomogeneous determinant, the two (equivalent) equalities

$$\chi_{D^*} \circ \rho_D^* = \chi_{f^*} = \chi_f^{-1} = \chi_D^{-1}, \tag{3.9}$$
$$d\chi_{D^*} \circ d\rho_D^* = d\chi_{f^*} = -d\chi_f = -d\chi_D$$

follow from Proposition 3.1 and (3.3).

The following observation was the starting point for our study of the symmetry of b-functions of prehomogeneous determinants in the remainder of this section.

Theorem 3.1. *M. Sato's fundamental theorem of homogeneous vector spaces as formulated in [Kim03, Prop. 4.18] holds true for reductive linear free divisors.*

Proof. The hypotheses for the proof of this theorem in [Kim03, §4.1] are fulfilled by Theorem 2.2.(c) except for the requirement that f be irreducible. But this latter assumption is used in the proof only to ensure that f and f^* can be chosen defined over $\mathbb{R}$ (see [Kim03, Prop. 4.1.(1), Prop. 4.2]) and that $|\det \rho(g)| = |\chi_f(g)|^{n/d}$ for all $g \in G$ where $d = \deg f$ (see [Kim03, p. 119]). But, by Theorem 2.1 and 2.2.(b), these statements hold true for any reductive linear free divisor. Indeed, it follows from Theorem 2.2.(a) that the Lie algebra representation $d\rho_{D^*}: \mathfrak{g}_{D^*} = d\rho_D^*(\mathfrak{g}_D) \hookrightarrow \mathfrak{gl}_{V^*}$ is defined over $\mathbb{Q}$ with respect

to $V_{\mathbb{Q}}^*$. Then the defining equation f^* of the dual linear free divisor D^* is defined over $\mathbb{Q}$ with respect to $V_{\mathbb{Q}}^*$ by Theorem 2.2.(b). Note also that the non-degeneracy of f and f^* used in [Kim03, Cor. 4.4] follows from [Kim03, Prop. 2.24] even in the reducible case. $\qquad\square$

A symmetry property of the b-function as in Lemma 3.1 below is a corollary of Sato's fundamental theorem (see [Kim03, Prop. 4.19]) for irreducible reductive prehomogeneous vector spaces. For reductive prehomogeneous determinants, we shall give an independent proof which is more elementary in this setting. Our argument also applies to regular linear free divisors (see Section 5).

Lemma 3.1. *Let D be a prehomogeneous determinant with $D^* \neq V^*$ such that* (3.4), (3.5), (2.12), *and* (3.9) *hold true. Then its b-function satisfies*

$$b_D(s) = (-1)^n \cdot b_D(-s-2).$$

Proof. From (2.6), (2.12), and (3.9) we deduce that

$$Q_A = \delta_A - s \cdot \operatorname{tr}(A) = \delta_A - s \cdot \delta_A(f)/f \in \operatorname{Ann}_{D_V[s]} f^s, \qquad (3.10)$$
$$Q_A^* = \delta_A^* + s \cdot \operatorname{tr}(A) = \delta_A^* - s \cdot \delta_A^*(f^*)/f^* \in \operatorname{Ann}_{D_{V^*}[s]}(f^*)^s,$$

for $A \in \mathfrak{g}_D$ when identifying $A = d\rho_D(A)$. So by (3.4) and (3.10), we have

$$P = f^*(\partial_x) \cdot f(x) - b_D(s) \in \operatorname{Ann}_{D_V[s]} f^s \ni Q_{A_k} = Q_k \qquad (3.11)$$

for $k = 1, \ldots, n$. Now let $\mathscr{D}_V$ denote the sheaf of algebraic differential operators on V. Then, at any point $p \in U = V \backslash D$, we have

$$\operatorname{Ann}_{\mathscr{D}_{V,p}[s]} f^s = \langle Q_1, \ldots, Q_n \rangle_{\mathscr{D}_{V,p}[s]}. \qquad (3.12)$$

Indeed, as $f \in \mathscr{O}_{V,p}^*$, conjugating by f^s reduces this statement to $\operatorname{Ann}_{\mathscr{D}_{V,p}[s]} 1 = \langle \delta_1, \ldots, \delta_n \rangle_{\mathscr{D}_{V,p}[s]}$ which holds true since $\langle \delta_1, \ldots, \delta_n \rangle_{\mathscr{O}_{V,p}} = \langle \partial_{x_1}, \ldots, \partial_{x_n} \rangle_{\mathscr{O}_{V,p}}$. The equality (3.12) gives rise to an exact sequence

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathscr{D}_U[s]^n \xrightarrow{\ (Q_1, \ldots, Q_n)\ } \operatorname{Ann}_{\mathscr{D}_U[s]} f^s \longrightarrow 0\ .$$

As U is affine and the sheaf $\mathscr{R}$ is quasicoherent, we have $H^1(U, \mathscr{R}) = 0$ (see [Meb89, Prop. 2.5.4] and [Har77, Thm. 3.5]) and this implies with (3.11) that

$$f^m \cdot P \in \langle Q_1, \ldots, Q_n \rangle_{D_V[s]}, \qquad (3.13)$$

for some $m \in \mathbb{N}$. Let $\mathscr{F}$ denote the Fourier transform. Then, by (2.5), (3.8), and (3.10), we have

$$
\begin{aligned}
\mathscr{F}(Q_A(s)) &= \mathscr{F}(x^t A^t \partial_x - s \cdot \operatorname{tr}(A)) \qquad (3.14) \\
&= -\partial_y^t A^t y - s \cdot \operatorname{tr}(A) \\
&= -y^t A \partial_y - (s+1) \cdot \operatorname{tr}(A) \\
&= \delta_A^* + (-s-1) \cdot \operatorname{tr}(A) = Q_A^*(-s-1).
\end{aligned}
$$

Combining (3.10), (3.13), and (3.14) yields that

$$
\begin{aligned}
\operatorname{Ann}_{D_{V^*}[s]}(f^*)^{-s-1} \ni \mathscr{F}(f^m \cdot P) &= \mathscr{F}(f(x))^m \cdot \mathscr{F}(f^*(\partial_x) \cdot f(x) - b_D(s)) \\
&= (-1)^{nm} f(\partial_y)^m \cdot ((-1)^n \cdot f^*(y) \cdot f(\partial_y) - b_D(s))
\end{aligned}
$$

and hence, using (3.4) and (3.5),

$$
\begin{aligned}
0 &= f(\partial_y)^m \cdot ((-1)^n \cdot f^*(y) \cdot f(\partial_y) - b_D(s)) \cdot (f^*)^{-s-1} \\
&= f(\partial_y)^m \cdot ((-1)^n \cdot b_{D^*}(-s-2) - b_D(s)) \cdot (f^*)^{-s-1} \\
&= ((-1)^n \cdot b_{D^*}(-s-2) - b_D(s)) \cdot f(\partial_y)^m \cdot (f^*)^{-s-1} \\
&= ((-1)^n \cdot b_D(-s-2) - b_D(s)) \cdot b_{D^*}(-s-2) \cdots b_{D^*}(-s-m-1) \cdot (f^*)^{-s-m-1}.
\end{aligned}
$$

Thus, $(-1)^n \cdot b_D(-s-2) - b_D(s) = 0$ as claimed. $\qquad \square$

We conclude the following from Lemma 3.1 using Theorem 2.1, and from (3.6), (3.7), and [Kas77].

Theorem 3.2. *For any reductive prehomogeneous determinant $D \subseteq V$ in dimension $\dim V = n$, the b-function $B_D(s) \equiv b_D(s)$ has degree n and rational negative roots symmetric about -1. In particular, -1 is the only integer root.*

Remark. In [CJUE04, Prob. 3.2], Castro-Jiménez and Ucha-Enríquez ask whether -1 is the only integer root of the b-function for any linear free divisor. Theorem 3.2 gives a positive answer to this question in the case of reductive prehomogeneous determinants. In Theorem 5.2 in Section 5, we shall also settle the case of certain non-reductive linear free divisors.

§4. Examples

Up to dimension 4 all linear free divisors have been classified in [GMNRS09, §6.4]. Table 1 lists their b-functions computed using `Macaulay 2` [M2]. Here

$\mathfrak{g}_2$ is the non-Abelian Lie algebra of dimension 2 and $\mathfrak{g}_3$ is characterized as having 2-dimensional Abelian derived algebra $\mathfrak{g}_3'$, on which the adjoint action of a basis vector outside $\mathfrak{g}_3'$ is semi-simple with eigenvalues 1 and 2 (see [Jac79, Ch. I, §4]). Observe that, except for having degree greater than n, also the non-reductive divisors in Table 1 satisfy the statement of Theorem 3.2, which supports Conjecture 1.2.

Table 1: b-functions of linear free divisors in dimension $n \leq 4$

n	f	$\mathfrak{g}_D$	reductive?	Spec $b(s)$
1	x	$\mathbb{C}$	Yes	-1
2	xy	$\mathbb{C}^2$	Yes	$-1, -1$
3	xyz	$\mathbb{C}^3$	Yes	$-1, -1, -1$
3	$(y^2 + xz)z$	$\mathfrak{b}_2$	No	$-\frac{5}{4}, -1, -1, -\frac{3}{4}$
4	$xyzw$	$\mathbb{C}^4$	Yes	$-1, -1, -1, -1$
4	$(y^2 + xz)zw$	$\mathbb{C} \oplus \mathfrak{b}_2$	No	$-\frac{5}{4}, -1, -1, -1, -\frac{3}{4}$
4	$(yz + xw)zw$	$\mathbb{C}^2 \oplus \mathfrak{g}_2$	No	$-\frac{4}{3}, -1, -1, -1, -\frac{2}{3}$
4	$x(y^3 - 3xyz + 3x^2w)$	$\mathbb{C} \oplus \mathfrak{g}_3$	No	$-\frac{7}{5}, -\frac{4}{3}, -\frac{6}{5},$ $-1, -1, -1,$ $-\frac{4}{5}, -\frac{2}{3}, -\frac{3}{5}$
4	$y^2z^2 - 4xz^3 - 4y^3w +$ $18xyzw - 27w^2x^2$	$\mathfrak{gl}_2(\mathbb{C})$	Yes	$-\frac{7}{6}, -1, -1, -\frac{5}{6}$

Discriminants of quiver representations are at the origin of linear free divisors and form a rich source of reductive determinants. We shall briefly review the basics of the theory to understand the following three examples and refer to [BM06] for more details.

A *quiver* $Q = (Q_0, Q_1)$ is a directed finite graph with vertex set Q_0 and edge set $Q_1 \subseteq Q_0 \times Q_0$. With a dimension vector $\mathbf{d} = (d_i)) \in \mathbb{N}^{Q_0}$ we associate the *representation space*

$$\mathrm{Rep}(Q, \mathbf{d}) = \prod_{(i,j) \in Q_1} \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}).$$

The *quiver group*

$$\mathrm{GL}(Q, \mathbf{d}) = \prod_{i \in Q_0} \mathrm{GL}_{d_i}(\mathbb{C})$$

acts on $\mathrm{Rep}(Q, \mathbf{d})$ through the *quiver representation*

$$\rho_{Q,\mathbf{d}} \colon \mathrm{GL}(Q, \mathbf{d}) \to \mathrm{GL}(\mathrm{Rep}(Q, \mathbf{d})),$$

$$\rho_{Q,\mathbf{d}}((g_i)_{i \in Q_0})(\varphi_{(i,j)})_{(i,j) \in Q_1} = (g_j \circ \varphi_{(i,j)} \circ g_i^{-1})_{(i,j) \in Q_1}.$$

Note that $Z = \mathbb{C} \cdot (I_{d_i})_{i \in Q_0}$, where $I_i \in \mathrm{GL}_i(\mathbb{C})$ denotes the unit matrix, lies in the kernel of $\rho_{Q,\mathbf{d}}$. The *Tits form* $q \colon \mathbb{C}^{Q_0} \to \mathbb{C}$ is defined by
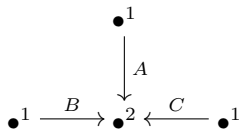
$$q((x_i)_{i \in Q_0}) = \sum_{i \in Q_0} x_i^2 - \sum_{(i,j) \in Q_1} x_i x_j.$$

The condition $\dim G = \dim V$ in Definition 2.1 for the discriminant D of $(G, \rho, V) = (\mathrm{GL}(Q, \mathbf{d})/Z, \rho_{Q,\mathbf{d}}, \mathrm{Rep}(Q, \mathbf{d}))$ to define a prehomogeneous determinant is equivalent to $q(\mathbf{d}) = 1$ and reductivity is automatic if $D \neq V$ [GMNRS09, §4].

The following example of a linear free divisor is the discriminant of the star quiver studied in [GMNRS09, Exa. 5.3]. We use the microlocal calculus developed in [SKKŌ81] to determine its b-function and then `Macaulay 2` [M2] to verify our result.

Example 1. Consider the *star quiver* Q shown in Figure 1 with one sink, three sources, and dimension vector $\mathbf{d} = (2, 1, 1, 1)$ with $q(\mathbf{d}) = 1$. The

Figure 1. The star quiver in Example 1

quiver group $G = \mathrm{GL}(Q, \mathbf{d}) = \mathrm{GL}_2(\mathbb{C}) \times (\mathbb{C}^*)^3 \subseteq \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$ acts on the 6-dimensional representation space $\mathrm{Rep}(Q, \mathbf{d}) = \mathbb{C}^{2 \times 3} = \{(A\ B\ C)\}$ of 2×3-matrices through $\rho = \rho_{Q,\mathbf{d}}$ given by

$$\rho(g)x = g_1 \cdot x \cdot g_2^{-1}, \quad g = (g_1, g_2).$$

The infinitesimal vector fields for this group action are given by

$$\xi_{i,j} = x_{i,1}\partial_{j,1} + x_{i,2}\partial_{j,2} + x_{i,3}\partial_{j,3}, \quad \xi_k = x_{1,k}\partial_{1,k} + x_{2,k}\partial_{2,k} \tag{4.1}$$

where $x_{i,j}$ are coordinates on $\mathbb{C}^{2 \times 3}$ and $\partial_{i,j} = \partial_{x_{i,j}}$. The discriminant is a reductive linear free divisor D defined by the product of maximal minors

$$f(x) = (x_{1,2}x_{2,3} - x_{2,2}x_{1,3})(x_{1,1}x_{2,3} - x_{2,1}x_{1,3})(x_{1,1}x_{2,2} - x_{2,1}x_{1,2})$$

with associated character

$$\chi(g) = \det(g_1)^3 \det(g_2)^{-2}. \tag{4.2}$$

We first enumerate the different orbits by specifying a generic element x_0. We denote by $O_{i,j}$ (or O_i if there is only one) an orbit of dimension i of a point $x_0^{i,j}$ (or x_0^j), and by $\Lambda_{i,j}$ (or Λ_i) its conormal space. In order to calculate the dimension of the orbit of x_0 it is sufficient to determine the tangent space $T_{x_0} = d\rho(\mathfrak{g})x_0$ which is spanned by the evaluation of the vector fields (4.1) at x_0.

1. Orbits of rank 2:

(1.a) 1 open orbit:

$$x_0^6 = \begin{pmatrix} 1\,0\,1 \\ 0\,1\,1 \end{pmatrix}$$

(1.b) 3 orbits of rank 2 and dimension 5 with all columns nonzero:

$$x_0^{5,1} = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,1 \end{pmatrix}, \quad x_0^{5,2} = \begin{pmatrix} 0\,1\,0 \\ 1\,0\,1 \end{pmatrix}, \quad x_0^{5,3} = \begin{pmatrix} 0\,0\,1 \\ 1\,1\,0 \end{pmatrix}.$$

(1.c) 3 orbits of rank 2 and dimension 4 with exactly one zero column:

$$x_0^{4,1} = \begin{pmatrix} 0\,1\,0 \\ 0\,0\,1 \end{pmatrix}, \quad x_0^{4,2} = \begin{pmatrix} 1\,0\,0 \\ 0\,0\,1 \end{pmatrix}, \quad x_0^{4,3} = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \end{pmatrix}.$$

2. Orbits of rank 1:

(2.a) 1 orbits of rank 1 and dimension 4:

$$x_0^{4,0} = \begin{pmatrix} 1\,1\,1 \\ 0\,0\,0 \end{pmatrix}$$

(2.b) 3 orbits of rank 1 and dimension 3:

$$x_0^{3,1} = \begin{pmatrix} 0\,1\,1 \\ 0\,0\,0 \end{pmatrix}, \quad x_0^{3,2} = \begin{pmatrix} 1\,0\,1 \\ 0\,0\,0 \end{pmatrix}, \quad x_0^{3,3} = \begin{pmatrix} 1\,1\,0 \\ 0\,0\,0 \end{pmatrix}.$$

(2.c) 3 orbits of rank 1 and dimension 2:

$$x_0^{2,1} = \begin{pmatrix} 1\,0\,0 \\ 0\,0\,0 \end{pmatrix}, \quad x_0^{2,2} = \begin{pmatrix} 0\,1\,0 \\ 0\,0\,0 \end{pmatrix}, \quad x_0^{2,3} = \begin{pmatrix} 0\,0\,1 \\ 0\,0\,0 \end{pmatrix}.$$

3. 1 Orbit of rank 0: $x_0^0 = 0$.

For each of the above points $x_0 = x_0^{i,j}$, we shall now compute the action ρ_{x_0} of the isotropy group G_{x_0} on the normal space $V_{x_0} = V/T_{x_0}$ to the tangent space T_{x_0} to the orbit of x_0. We omit the trivial cases of the open and of the zero orbit and describe V_{x_0} by identifying it with a representative subspace in V.

1. $x_0 = x_0^{5,1}$:

$$G_{x_0} = \left\{ g = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \times (a,1,1) \right\}, \quad T_{x_0} = \langle \partial_{1,1}, \partial_{2,1}, \partial_{2,2}, \partial_{2,3}, \partial_{1,2} + \partial_{1,3} \rangle,$$

$$V_{x_0} = \left\{ x = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad \rho_{x_0}(g)x = a \cdot x.$$

2. $x_0 = x_0^{4,3}$:

$$G_{x_0} = \left\{ g = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \times (1,a,b) \right\}, \quad T_{x_0} = \langle \partial_{1,1}, \partial_{1,2}, \partial_{2,1}, \partial_{2,2} \rangle,$$

$$V_{x_0} = \left\{ x = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & v \end{pmatrix} \right\}, \quad \rho_{x_0}(g)x = \frac{1}{b} \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & av \end{pmatrix}.$$

3. $x_0 = x_0^{4,0}$:

$$G_{x_0} = \left\{ g = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \right\}, \quad T_{x_0} = \langle \partial_{1,1}, \partial_{1,2}, \partial_{1,3}, \partial_{2,1} + \partial_{2,2} + \partial_{2,3} \rangle,$$

$$V_{x_0} = \left\{ x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & u & v \end{pmatrix} \right\}, \quad \rho_{x_0}(g)x = b \cdot x.$$

This is the only non-prehomogeneous normal action.

4. $x_0 = x_0^{3,3}$:

$$G_{x_0} = \left\{ g = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \times (1,1,c) \right\}, \quad T_{x_0} = \langle \partial_{1,1}, \partial_{1,2}, \partial_{2,1} + \partial_{2,2} \rangle,$$

$$V_{x_0} = \left\{ x = \begin{pmatrix} 0 & 0 & u \\ 0 & v & w \end{pmatrix} \right\}, \quad \rho_{x_0}(g)x = \begin{pmatrix} 0 & \frac{u+aw}{c} \\ bv & b\frac{w}{c} \end{pmatrix}.$$

This is a prehomogeneous action with non-reduced discriminant $\{vw^2 = 0\}$.
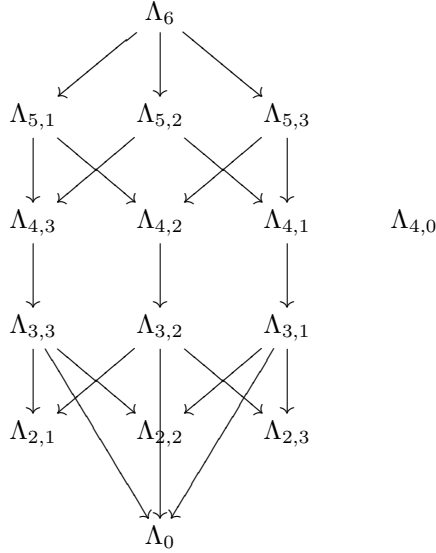
5. $x_0 = O_{2,1}$:

$$G_{x_0} = \left\{ g = \begin{pmatrix} 1\ a \\ 0\ b \end{pmatrix} \times (1, c, d) \right\}, \quad T_{x_0} = \langle \partial_{11}, \partial_{21} \rangle,$$

$$V_{x_0} = \left\{ x = \begin{pmatrix} 0\ r\ s \\ 0\ t\ u \end{pmatrix} \right\}, \quad \rho_{x_0}(g)x = \begin{pmatrix} 0 & \frac{r+at}{c} & \frac{s+au}{d} \\ 0 & \frac{bt}{c} & \frac{bu}{d} \end{pmatrix}$$

This is a prehomogeneous action with reduced discriminant $\{tu(ur - st) = 0\}$

The holonomy diagram in Figure 2 shows the incidence relations between the good orbit conormals subject to intersections in codimension one. Recall that $\Lambda_{4,0}$ is not good as it is not prehomogeneous. We shall apply the method

Figure 2. Holonomy diagram of Example 1

in [SKKŌ81] to the chain

$$\Lambda_6 \longrightarrow \Lambda_{5,1} \longrightarrow \Lambda_{4,3} \longrightarrow \Lambda_{3,3} \longrightarrow \Lambda_0. \tag{4.3}$$

An elementary local calculation shows that all the intersections in this chain are smooth and transversal at the generic point which is the situation of [SKKŌ81, Cor. 7.6]. Therefore it remains to calculate $\operatorname{ord}_\Lambda f^s$ for all Λ in the chain (4.3)

using the following formula from [SKKŌ81, Prop. 4.14]:

$$\mathrm{ord}_\Lambda f^s = -m_\Lambda s - \frac{\mu_\Lambda}{2} = d\chi(A_0) - \mathrm{tr}_{V_{x_0}^*}\, d\rho_{x_0}^*(A_0) + \frac{1}{2}\dim V_{x_0}^*. \qquad (4.4)$$

Here $A_0 = (A_0^1, A_0^2) \in \mathfrak{g}_{x_0}$ and $(x_0, y_0) \in \Lambda$ is generic such that $d\rho_{x_0}(A_0)y_0 = y_0$. By (4.2), we have

$$d\chi(A_0) = 3 \cdot \mathrm{tr}(A_0^1) - 2 \cdot \mathrm{tr}(A_0^2), \quad \mathrm{tr}_{V_{x_0}^*}\, d\rho_{x_0}^*(A_0) = -\mathrm{tr}_{V_{x_0}}\, d\rho_{x_0}(A_0)$$

which can be determined from the preceding computations. Note that the contragredient action ρ^* on the dual space $(\mathbb{C}^{2\times 3})^*$ identified with $\mathbb{C}^{3\times 2}$ by the trace pairing is given by

$$d\rho^*(g)\xi = g_2\xi {g_1}^{-1}, \quad g = (g_1, g_2).$$

One computes

$$\mathrm{ord}_{\Lambda_6} f^s = 0, \quad \mathrm{ord}_{\Lambda_{5,1}} f^s = -s - \frac{1}{2}, \quad \mathrm{ord}_{\Lambda_{4,3}} f^s = -2s - 1,$$

$$\mathrm{ord}_{\Lambda_{3,3}} f^s = -5s - \frac{5}{2}, \quad \mathrm{ord}_{\Lambda_0} f^s = -6s - 3$$

Using the formula in [SKKŌ81, Cor. 7.6] this yields

$$\frac{b_{\Lambda_{5,1}}}{b_{\Lambda_6}} = s + 1, \quad \frac{b_{\Lambda_{4,3}}}{b_{\Lambda_{5,1}}} = -s - \frac{1}{2} + 2s + 1 + \frac{1}{2} = s + 1,$$

$$\frac{b_{\Lambda_{3,3}}}{b_{\Lambda_{4,3}}} = (3s+2)(3s+3)(3s+4), \quad \frac{b_{\Lambda_0}}{b_{\Lambda_{3,3}}} = -5s - \frac{5}{2} + 6s + 3 + \frac{1}{2} = s + 1.$$
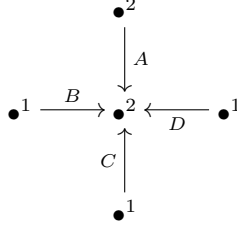
From this we conclude that

$$B_D(s) = b_{\Lambda_0}(s) = \left(s + \frac{2}{3}\right)(s+1)^4\left(s + \frac{4}{3}\right)$$

is the *b*-function of D which is also confirmed by `Macaulay 2` [M2].

The next two examples are reductive non-reduced prehomogeneous determinants arising from quiver representations.

Example 2. We modify the star quiver in Example 1 by adding an additional source. Consider the resulting star quiver $\widetilde{D}_3$ in Figure 3 and the dimension vector $\mathbf{d} = (2, 2, 1, 1, 1)$ with $q(\mathbf{d}) = 1$. The quiver group $\mathrm{GL}(Q, \mathbf{d}) = \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times (\mathbb{C}^*)^3$ acts on a 10-dimensional representation space $\mathrm{Rep}(Q, \mathbf{d}) = \mathbb{C}^{2\times 2} \times \mathbb{C}^{3\times 2} = \{(A, (B\ C\ D))\}$ and the orbit of

Figure 3. The star quiver in Example 2

$$\left(\begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix}, \begin{pmatrix} 1\,0\,1 \\ 0\,1\,1 \end{pmatrix} \right)$$ is open. The non-reduced discriminant D defined by the function

$$f(x) = (x_{1,1}x_{2,2} - x_{1,2}x_{2,1})^2 \cdot (x_{1,3}x_{2,4} - x_{1,4}x_{2,3}) \cdot (x_{1,3}x_{2,5} - x_{1,5}x_{2,3}) \cdot (x_{1,4}x_{2,5} - x_{1,5}x_{2,4})$$

is a reductive prehomogeneous determinant. It is a union $D = D_1 \cup D_2$ where D_1 and D_2 are defined by the functions

$$f_1(x) = (x_{1,1}x_{2,2} - x_{1,2}x_{2,1})^2, \tag{4.5}$$
$$f_2(x) = (x_{1,3}x_{2,4} - x_{1,4}x_{2,3}) \cdot (x_{1,3}x_{2,5} - x_{1,5}x_{2,3}) \cdot (x_{1,4}x_{2,5} - x_{1,5}x_{2,4}).$$

Since D_1 is the square of a quadratic form and D_2 is the reductive linear free divisor from Example 1, we have

$$B_{D_1}(s) = (2s+1)(s+1)^2(2s+3)$$
$$B_{D_2}(s) = \left(s + \frac{2}{3} \right)(s+1)^4 \left(s + \frac{4}{3} \right).$$

As the functions in (4.5) depend on separate sets of variables, $B_D(s)$ divides $B_{D_1}(s) \cdot B_{D_2}(s)$ and hence
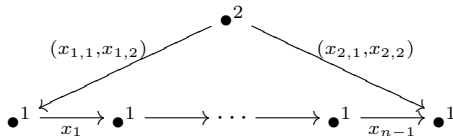
$$B_D = B_{D_1} \cdot B_{D_2} = \left(s + \frac{1}{2} \right) \left(s + \frac{2}{3} \right)(s+1)^6 \left(s + \frac{4}{3} \right) \left(s + \frac{3}{2} \right)$$

as $\deg B_D = 10 = \deg B_{D_1} + \deg B_{D_2}$ by Theorem 3.2. For the reduced discriminant D_{red}, we loose the symmetry property. Indeed, one computes

$$B_{D_{\mathrm{red}}}(s) = \left(s + \frac{2}{3} \right)(s+1)^5 \left(s + \frac{4}{3} \right)(s+2)$$

using the same degree argument.

Figure 4. The quiver in Example 3

Example 3. Consider the quiver representation associated with the quiver $\widetilde{A}_n$ in Figure 4 and the dimension vector $\mathbf{d} = (2, 1, \ldots, 1)$ with $q(\mathbf{d}) = 1$. The representation space is $(n+3)$-dimensional and the orbit of

$$\left(\begin{pmatrix} x_{1,1} \ x_{1,2} \\ x_{2,1} \ x_{2,2} \end{pmatrix}, (x_1, \ldots, x_{n-1}) \right) = \left(\begin{pmatrix} 1 \ 0 \\ 0 \ 1 \end{pmatrix}, (1, \ldots, 1) \right)$$

is open. The non-reduced discriminant D defined by the function

$$f(x) = \det \begin{pmatrix} x_{1,1} \ x_{1,2} \\ x_{2,1} \ x_{2,2} \end{pmatrix}^2 \cdot x_1 \cdots x_{n-1} = (x_{1,1} x_{2,2} - x_{1,2} x_{2,1})^2 \cdot x_1 \cdots x_{n-1}$$

is a reductive prehomogeneous determinant. As in Example 2, one computes

$$B_D(s) = \left(s + \frac{1}{2} \right) (s+1)^{n+1} \left(s + \frac{3}{2} \right),$$

$$B_{D_{\mathrm{red}}}(s) = (s+1)^n (s+2),$$

and the reduced discriminant does not have the symmetry property.

§5. b-functions of special linear free divisors

The symmetry of the b-functions of non-reductive prehomogeneous determinants in Section 4 motivates to try to weaken the hypotheses of Theorem 3.2. In the case of linear free divisors, this turns out to be possible.

Consider a linear free divisor D defined by f as in Definition 2.1. In [GMNRS09, Thm. 6.1], we proved that $\mathfrak{a}_D$ has a basis $A_2, \ldots, A_n = S_2, \ldots, S_s, N_1, \ldots, N_{n-r}$ as in the proof of Theorem 2.1: the $d\rho_D(S_i)$ are semisimple and commute, the $d\rho_D(N_j)$ are nilpotent, and the N_J are homogeneous with respect to the S_i with rational weights. Then, as in (2.13), we have

$$d\chi_f(A_i) = \mathrm{tr} \circ d\rho_D(A_i) + \sum_{j=1}^{n} [A_i, A_j]/A_j, \quad i = 1, \ldots, s. \qquad (5.1)$$

Assume now that $D^* \neq V^*$. Then the analog of (5.1) for D^* reads

$$d\chi_{f^*}(A_i) = \operatorname{tr} \circ d\rho_D^*(A_i) + \sum_{j=1}^{n}[A_i, A_j]/A_j, \quad i = 1, \ldots, s. \qquad (5.2)$$

As $\operatorname{tr} \circ d\rho_D^*(A_i) = -\operatorname{tr} \circ d\rho_D(A_i)$, we obtain the equalities

$$d\chi_f(A_i) - d\chi_{f^*}(A_i) = 2\operatorname{tr} \circ d\rho_D(A_i), \quad i = 1, \ldots, n, \qquad (5.3)$$

as the difference of (5.1) and (5.2) for $i = 1, \ldots, s$. Note that both sides of (5.3) are zero for $i = s + 1, \ldots, n$. From (5.3), we conclude the following formulas.

Lemma 5.1.   *For a linear free divisor D with $D^* \neq V^*$,*

$$\chi_f \cdot \chi_{f^*}^{-1} = \det^2 \circ \rho_D, \quad d\chi_f - d\chi_{f^*} = 2\operatorname{tr} \circ d\rho_D.$$

$\square$

As an immediate consequence of Lemma 5.1, (2.12) and (3.9) are equivalent.

Proposition 5.1.   *For a linear free divisor D with $D^* \neq V^*$, we have $\chi_D = \det \circ \rho_D$ if and only if $\chi_{f^*} = \chi_f^{-1}$.* $\square$

A relation of (3.4), (3.5), and (3.9), can be established for general prehomogeneous determinants.

Proposition 5.2.   *Let D be a prehomogeneous determinant with $D^* \neq V^*$. Then (3.4) is equivalent to (3.9) and implies (3.5).*

Proof.   The classical proof essentially works in our setting:

The existence of $b_f(s)$ and b_{f^*} satisfying (3.4) follows from (3.9) by the character argument in [Kim03, Prop. 2.22] which also gives the converse implication.

In the reductive case,

$$f(x) = \sum_{|\alpha|=n} f_\alpha \cdot x^\alpha, \quad f^*(y) = \sum_{|\alpha|=n} f_\alpha^* \cdot x^\alpha$$

have complex conjugate coefficients $f_\alpha = \overline{f_\alpha^*}$ in unitary coordinates. This is not necessary to show that

$$b_f(0) = f^*(\partial_x) \cdot f(x) = \sum_{|\alpha|=n} f_\alpha \cdot f_\alpha^* \cdot \alpha! = f(\partial_y) \cdot f^*(y) = b_{f^*}(0)$$

and similarly

$$b_f(0) \cdots b_f(m-1) = f^*(\partial_x)^m \cdot f(x)^m = f(\partial_y)^m \cdot f^*(y)^m = b_{f^*}(0) \cdots b_{f^*}(m-1)$$

for any $m \in \mathbb{N}$. The equality in (3.5) follows as in [Kim03, Prop. 2.23]. $\square$

In [dGMS09, Def. 2.1], a linear free divisor is called *special* if (2.12) holds true. Using this terminology, we conclude from Lemma 3.1 and Propositions 5.1 and 5.3 the following result analogous to Theorem 3.2.

Theorem 5.1.   *Let $D \subseteq V$ be a special linear free divisor in dimension* $\dim V = n$ *with* $D^* \neq V^*$. *Then the b-function $b_D(s)$ has degree at most n and roots symmetric about -1.* $\square$

Under the stronger assumption of regularity we can prove more.

Theorem 5.2.   *Let $D \subseteq V$ be a regular special linear free divisor in dimension* $\dim V = n$. *Then the b-function $B_D(s) \equiv b_D(s)$ has degree at most n and negative rational roots symmetric about -1. In particular, -1 is the only integer root.*

Proof.   Note that the hypothesis $D^* \neq V^*$ in Theorem 5.1 becomes redundant for regular D by [Kim03, Thm. 2.16].

By [SKKŌ81, Prop. 4.6], regularity of D is equivalent to $V^* = T_0^* V$ being a good holonomic variety. Under this latter hypothesis the arguments in [Gyo91, Lem. 2.5.7] still yield (3.7). $\square$

Note that in Theorem 5.2 regularity of f is equivalent to $\deg b_D(s) = n$. Indeed, the degree n coefficient of $b_f(s)$ is given by $f^*(\operatorname{grad}\log f) \cdot f$ by [Kim03, Prop. 2.13.(2)] and $\operatorname{grad}\log f\colon V \to V^*$ is a G-equivariant map by [Kim03, Prop. 2.13.(1)].

§6.   Euler homogeneity and normal representations

The condition of strong Euler homogeneity arose in the context of the logarithmic comparison theorem in [CMMNMCJ02, Conj. 1.4] (see also [CJNMM96] and [GS06]) and has been studied in [GMNRS09, §7] in the case of linear free divisors. It is conceivable that all linear free divisors are strongly Euler homogeneous, but there is neither a proof nor a counter-example.

Definition 6.1.   Let $D \subseteq V$ be a reduced hypersurface. A (local) vector field $\epsilon \in \operatorname{Der}(-\log D)$ is called an *Euler vector field* for D if $\epsilon(f) = f$ for some

(local) defining equation f of D. It is called an *Euler vector field at $p \in D$* if in addition $\epsilon(p) = 0$. The hypersurface D is called *strongly Euler homogeneous at $p \in D$* if there is an Euler vector field at p defined locally at p. By *strong Euler homogeneity* of D we mean that this property holds for all $p \in D$.

The following statement is implicit in the proof of [GMNRS09, Lem. 7.5].

Lemma 6.1.   *If D is (locally) defined by f and Euler homogeneous with Euler vector field ϵ then D is strongly Euler homogeneous if $\epsilon(p) \in \operatorname{Der}(-\log f)(p)$ for all $p \in D$.*

Assume now that D is a linear free divisor. Then the preceding observation can be formulated as follows.

Lemma 6.2.   *Strong Euler homogeneity of of a linear free divisor D means that the G_D-orbits and A_D-orbits in D coincide.*

We shall now give a sufficient condition for strong Euler homogeneity of D at $x_0 \in D$. Let G_{D,x_0} be the stabilizer of x_0 in G_D with Lie algebra $\mathfrak{g}_{D,x_0}$ and consider the representation $\rho_{D,x_0} \colon G_{D,x_0} \to V_{x_0}$ induced by ρ_D on the normal space $V_{x_0} = V/d\rho_D(\mathfrak{g}_D)x_0$ to the tangent space of the G_D-orbit of x_0 at x_0.

The localization $f_{x_0} \in \operatorname{Sym}(V^*)$ of f at x_0 [SKKŌ81, Def. 6.7] is the homogeneous nonzero polynomial defined by

$$f(x_0 + \varepsilon x') \equiv \varepsilon^k \cdot f_{x_0}(x') \mod \langle \varepsilon^{k+1} \rangle.$$

By [SKKŌ81, Def. 6.8], $f_{x_0}(x')$ induces a relative invariant $f_{x_0}(y)$ of the representation $(G_{D,x_0}, \rho_{D,x_0}, V_{x_0})$ with character $\chi_{f_{x_0}} = \chi_D$ where $y = y_1, \ldots, y_k$ are coordinates on V_{x_0}. Note however that the latter might not be a prehomogeneous vector space.

In the context of linear free divisors, Definition 2.1 applied to $(G, \rho, V) = (G_{D,x_0}, \rho_{D,x_0}, V_{x_0})$, as in Definition 3.1, gives another natural candidate D_{x_0} for such an invariant.

Definition 6.2.   Assume that $A_1, \ldots, A_k$ form a basis of $\mathfrak{g}_{D,x_0}$ and define $D_{x_0} \subseteq V_{x_0}$ by

$$f'_{x_0}(y) = \det(d\rho_{D,x_0}(A_1)y, \ldots, d\rho_{D,x_0}(A_k)y) \in \operatorname{Sym}(V_{x_0}^*).$$

Then $f'_{x_0} \neq 0$ or $D_{x_0} \neq V_{x_0}$ is equivalent to $(G_{D,x_0}, \rho_{D,x_0}, V_{x_0})$ being a prehomogeneous vector space.

Lemma 6.3. *Let D be a linear free divisor and assume that $x_0 \in D$ such that $D_{x_0} \neq V_{x_0}$. Then $f'_{x_0}(y) = f_{x_0}(y)$.*

Proof. Note that $A_{k+1}x_0, \ldots, A_n x_0$ form a basis of the tangent space $d\rho_D(\mathfrak{g}_D)x_0$ to the G_D-orbit of x_0 at x_0. We may assume that $d\rho_D(\mathfrak{g}_D)x_0 = 0 \times \mathbb{C}^{n-k}$ and then that

$$(A_{k+1}x_0, \ldots, A_n x_0) = \begin{pmatrix} 0 \ldots 0 \\ \vdots \quad \vdots \\ 0 \ldots 0 \\ 1 \ldots 0 \\ \vdots \ddots \vdots \\ 0 \ldots 1 \end{pmatrix} \tag{6.1}$$

We may then choose $y = x'_1, \ldots, x'_k$. Using (2.1) and (6.1), we compute

$$f(x_0 + \varepsilon x') = \det(A_1(x_0 + \varepsilon x'), \ldots, A_n(x_0 + \varepsilon x')) \tag{6.2}$$
$$\equiv \varepsilon^k \cdot \det(A_1 x', \ldots, A_k x', A_{k+1}x_0, \ldots, A_n x_0) \mod \langle \varepsilon^{k+1} \rangle.$$

Denoting by A'_i the matrix consisting of the upper k rows of A_i, (6.1) implies that

$$\det(A_1 x', \ldots, A_k x', A_{k+1}x_0, \ldots, A_n x_0) = \det(A'_1 x', \ldots, A'_k x') \tag{6.3}$$
$$\equiv \det(d\rho_{D,x_0}(A_1)y, \ldots, d\rho_{D,x_0}(A_k)y)$$
$$= f'_{x_0}(y) \mod \langle x_{k+1}, \ldots, x_n \rangle$$

By hypothesis, $f'_{x_0} \neq 0$ and the claim follows from (6.2) and (6.3). $\qquad\square$

Proposition 6.1. *Let D be a linear free divisor and assume that $x_0 \in D$ such that $D_{x_0} \neq V_{x_0}$. Then D is strongly Euler homogeneous at x_0.*

Proof. By assumption $(G_{D,x_0}, \rho_{D,x_0}, V_{x_0})$ is a prehomogeneous vector space and $f_{x_0}(y)$ is a nonzero relative invariant with character χ_D. Thus, there must be an infinitesimal vector field as in (2.5) that is not tangent to the level sets of $f_{x_0}(y)$. By (2.6), this means that $\chi_D(B) = 1$ for some $B \in \mathfrak{g}_{x_0}$. Using (2.2) and (2.4) we conclude that $B = A_1 - A$ with $A \in \mathfrak{a}_D$ and hence $\rho_D(A_1)x_0 = \rho_D(A)x_0$ by definition of $\mathfrak{g}_{x_0}$. Then the claim follows from Lemma 6.1. $\qquad\square$

§7. Koszul freeness and Euler homogeneity

Koszul-freeness is a natural finiteness condition on logarithmic vector fields and is also closely related to the logarithmic comparison theorem (see [Tor04] and [CMNM05]). In geometric terms, *Koszul freeness* of a free divisor $D \subseteq V$ can be defined as the local finiteness of the *logarithmic stratification* by maximal integral submanifolds along logarithmic vector fields as defined in [Sai80, §3]. As shown in [GMNRS09, Thm. 7.4], this condition can be formulated algebraically as the symbols of a basis of $\mathrm{Der}(-\log D) \subseteq \mathscr{D}_V$ forming a regular sequence in $\mathrm{gr}_F \mathscr{D}_V = \mathscr{O}_V \otimes \mathrm{Sym}(V)$ where F denotes the order filtration on $\mathscr{D}_V$. Avoiding the technical details, this equivalence can be understood as follows: At each point $p \in V$, $\mathrm{Der}(-\log D)(p)$ spans the tangent space of the logarithmic stratum through p. Thus, the *logarithmic variety* in T^*V defined by the symbols of local bases of $\mathrm{Der}(-\log D)$ is the union of conormals to the logarithmic strata (see [Sai80, (3.16)]). It is of (co-)dimension n exactly where the symbols of n local basis elements of $\mathrm{Der}(-\log D)$ form a regular sequence in the Cohen–Macaulay coordinate ring $\mathscr{O}_V \otimes \mathrm{Sym}(V)$ of T^*V (see [CMNM02, 1.8]).

It is tempting to introduce a stronger version of Koszul freeness by imposing local finiteness of the logarithmic stratification of $\mathrm{Der}(-\log f)$ on D. But unless D is strongly Euler homogeneous, which reduces this condition to ordinary Koszul freeness by Lemma 6.1, it is not clear that the condition in question depends only on D.

For a linear free divisor D, the logarithmic stratification is the stratification by orbits of G_D consisting of smooth locally closed algebraic varieties (see [Hum75, Prop. 8.3]). Using the action of $\mathbb{C}^* \subseteq G_D$, one can see that local finiteness of this stratification is equivalent to finiteness. Moreover, there is a natural choice of defining equation which we use to introduce the desired strong version of Koszul freeness for this class of divisors.

Definition 7.1. We call a linear free divisor D *Koszul free* if the stratification of D by orbits of G_D is finite, and *strongly Koszul free*, if the same holds for the stratification by orbits of A_D.

Note that the G_D-orbit $V \backslash D$ is a union of infinitely many A_D-orbits by reasons of dimension. The algebraic definition of Koszul freeness above translates to strong Koszul freeness as follows.

Proposition 7.1. *A linear free divisor D is Koszul free if and only if the symbols of a basis of $\mathfrak{g}_D$ form a regular sequence in $\mathrm{gr}_F D_V$. It is strongly Koszul free if and only if the symbols of a basis of $\mathfrak{a}_D$ and f form a regular sequence in $\mathrm{gr}_F D_V$.*

Proof. Let $\delta_2, \ldots, \delta_n$ be a basis of $\mathfrak{a}_D$ and denote by $\sigma_F \colon D_V \to \operatorname{gr}_F D_V$ the symbol map for the order filtration F on D_V. As $\operatorname{gr}_F D_V = \operatorname{Sym}(V^* \oplus V)$ is a polynomial ring hence Cohen–Macaulay, we have to show that the variety $\operatorname{Var}(f, \sigma_F(\delta_2), \ldots, \sigma_F(\delta_n))$ has (co-)dimension n exactly if the stratification of D by orbits of A_D is finite. As $\mathfrak{a}_D$ spans the tangent spaces of the orbits, $\operatorname{Var}(\sigma_F(\delta_2), \ldots, \sigma_F(\delta_n))$ is the union of conormals to the orbits and $\operatorname{Var}(f) = T^*V|_D$. The second claim follows and the proof of the first one is similar. $\square$

Remark. The following generalization of strong Koszul freeness to arbitrary free divisors has been suggested to us by L. Narváez Macarro: Let $D \subseteq X$ be a free divisor in a complex manifold. Then we call D *strongly Koszul free* if, locally at any point $p \in D$, and for some (and hence any) reduced local defining equation f of D at p and for some (and hence any) local basis $\delta_1, \ldots, \delta_n$ of $\operatorname{Der}(-\log D)$, the symbols of f and $\delta_i - \delta_i(f)/f \cdot s$, $i = 1 \ldots, n$, with respect to the (total) order filtration F form a regular sequence in $\operatorname{gr}_F \mathscr{D}_{X,p}[s] = \operatorname{Sym}_{\mathscr{O}_{X,p}} F^1 \mathscr{D}_{X,p}$. By [NM08, Prop. 2.2.16] and Proposition 7.1, this reduces to Definition 7.1 in the special case. Using ideas of A. Simis and T. Torrelli [CMNM09, Prop. 1.27], Narváez Macarro can prove that strong Koszul freeness is equivalent to being free with Jacobian ideal of linear type.

Let us now relate the properties of strong Euler homogeneity, Koszul freeness, and strong Koszul freeness for linear free divisors.

Proposition 7.2. *The following implications hold for any linear free divisor D.*

(a) *Strong Koszul freeness implies Koszul freeness.*

(b) *Strong Koszul freeness implies strong Euler homogeneity.*

(c) *For Koszul free D, strong Euler homogeneity and strong Koszul freeness are equivalent.*

Proof. (a) is clear by Definition 7.1 and (c) follows from (b) and Lemma 6.2.
For $0 \leq k \leq n - 1$, let $T_k = \{p \in D \mid \operatorname{rk} \mathfrak{a}_D(p) \leq k\}$ (see [GMNRS09, §7] and [Sai80, Def. 3.12]). Then T_k is a homogeneous algebraic variety defined by f and the $(k+1) \times (k+1)$-minors of the matrix $(\delta_i)_{i=2,\ldots,n}$ and equals the union of A_D-orbits of dimension at most k. As the Euler vector field ϵ is the infinitesimal generator of the $\mathbb{C}^*$-action that defines homogeneity, ϵ is tangent to T_k and to all G_D-orbits. Therefore, T_k is also a union of G_D-orbits. So

if D is strongly Koszul free, $T_k \backslash T_{k-1}$ is the finite union of all k-dimensional A_D-orbits each of which is also a G_D-orbit. This proves (b).   □

Finally, we are ready to prove our main application to the LCT problem.

Proof of Theorem 1.4.   Let D be a reductive linear Koszul free divisor defined by f. Because of the $\mathbb{C}^*$-action, the (global) b-function of D coincides with the local b-function of the germ f_0 of f at the origin (see [Gyo91, Lems. 2.5.3-2.5.4]). By Proposition 7.1 and flatness of $\mathscr{O}_V$ over $\mathrm{Sym}(V^*)$, Koszul freeness of a linear free divisor D is equivalent to Koszul freeness of $(f, 0)$. By the same argument, strong Koszul freeness of D is equivalent to condition (c) in Theorem 1.2 for f_0. Therefore, the claim follows by combining Theorem 3.2 and Proposition 7.2.(c) with Theorems 1.1 and 1.2.   □

Acknowledgments

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