A RESIDUAL DUALITY OVER GORENSTEIN RINGS WITH APPLICATION TO LOGARITHMIC DIFFERENTIAL FORMS

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ABSTRACT. Kyoji Saito's notion of a free divisor was generalized by the first author to reduced Gorenstein spaces and by Delphine Pol to reduced Cohen–Macaulay spaces. Starting point is the Aleksandrov–Terao theorem: A hypersurface is free if and only if its Jacobian ideal is maximal Cohen–Macaulay. Pol obtains a generalized Jacobian ideal as a cokernel by dualizing Aleksandrov's multi-logarithmic residue sequence. Notably it is essentially a suitably chosen complete intersection ideal that is used for dualizing. Pol shows that this generalized Jacobian ideal is maximal Cohen–Macaulay if and only if the module of Aleksandrov's multi-logarithmic differential k-forms has (minimal) projective dimension $k-1$, where k is the codimension in a smooth ambient space. This equivalent characterization reduces to Saito's definition of freeness in case $k=1$. In this article we translate Pol's duality result in terms of general commutative algebra. It yields a more conceptual proof of Pol's result and a generalization involving higher multi-logarithmic forms and generalized Jacobian modules.

CONTENTS

1. Introduction 1
Acknowledgments 4
2. Residual duality over Gorenstein rings 4
2.1. I-duality and I-free approximation 5
2.2. I-reflexivity over Gorenstein rings 8
2.3. R-dual I-free approximation 11
2.4. Projective dimension and residual depth 14
2.5. Restricted I-free approximation 15
3. Application to logarithmic forms 18
3.1. Log forms along complete intersections 19
3.2. Log forms along Cohen–Macaulay spaces 22
3.3. Higher log vector fields and Jacobian modules 23
References 26

1. INTRODUCTION

Logarithmic differential forms along hypersurfaces and their residues were introduced by Kyoji Saito (see [Sai80]). They are part of his theory of primitive forms and period mappings where the hypersurface is the discriminant of a universal unfolding of a function with isolated critical point (see [Sai81, Sai83]). The special case of normal crossing divisors appeared earlier in Deligne's construction of mixed Hodge structures (see [Del71]). Here the logarithmic differential 1-forms form a locally free sheaf. In general a divisor with this property is called a free divisor. Further examples include plane curves (see [Sai80,

2010 *Mathematics Subject Classification.* Primary 13H10; Secondary 13C14, 32A27.
Key words and phrases. Duality, Gorenstein, logarithmic differential form, residue, free divisor.

(1.7)]), unitary reflection arrangements and their discriminants (see [Ter83, Thm. C]) and discriminants of versal deformations of isolated complete intersection singularities and space curves (see [Loo84, (6.13)] and [vS95]). Free divisors also occur as discriminants in prehomogeneous vector spaces (see [GMS11]). In case of hyperplane arrangements the study of freeness attracted a lot of attention (see [Yos14]).

Let D be a germ of reduced hypersurface in $Y \cong (\mathbb{C}^n, 0)$ defined by $h \in \mathscr{O}_Y$. The $\mathscr{O}_Y$-modules $\Omega^q(\log D)$ of logarithmic differential q-forms along D and the $\mathscr{O}_D$-modules ω_D^p of regular meromorphic differential p-forms on D fit into a short exact logarithmic residue sequence (see [Sai80, §2] and [Ale88, §4])

$$0 \longrightarrow \Omega_Y^q \longrightarrow \Omega^q(\log D) \xrightarrow{\operatorname{res}_D^q} \omega_D^{q-1} \longrightarrow 0.$$

Denoting by $\nu_D \colon \widetilde{D} \to D$ the normalization of D, $(\nu_D)_*\mathscr{O}_{\widetilde{D}} \subseteq \omega_D^0$ (see [Sai80, (2.8)]). For plane curves Saito showed that equality holds exactly for normal crossing curves (see [Sai80, (2.11)]). Granger and the first author (see [GS14]) generalized this fact and thus extended the Lê–Saito Theorem (see [LS84]) by an equivalent algebraic property. They showed that $(\nu_D)_*\mathscr{O}_{\widetilde{D}} = \omega_D^0$ if and only if D is normal crossing in codimension one, that is, outside of an analytic subset of Y of codimension at least 3. The proof uses the short exact sequence

$$0 \longleftarrow \mathscr{J}_D \xleftarrow{\langle -, dh \rangle} \Theta_Y \longleftarrow \operatorname{Der}(-\log D) \longleftarrow 0$$

obtained as the $\mathscr{O}_Y$-dual of the logarithmic residue sequence. It involves the Jacobian ideal $\mathscr{J}_D$ of D, the $\mathscr{O}_Y$-module $\Theta_Y := \operatorname{Der}_{\mathbb{C}}(\mathscr{O}_Y) \cong (\Omega_Y^1)^*$ of vector fields on Y and its submodule $\operatorname{Der}(-\log D) \cong \Omega^1(\log D)^*$ of logarithmic vector fields along D. It is shown that $\omega_D^0 = \mathscr{J}_D^*$ and that $\mathscr{J}_D = (\omega_D^0)^*$ if D is a free divisor. In fact freeness of D is equivalent to $\mathscr{J}_D$ being a Cohen–Macaulay ideal by the Aleksandrov–Terao theorem (see [Ale88, §2] and [Ter80, §2]).

As observed by first author (see [Sch16]) the inclusion $(\nu_D)_*\mathscr{O}_{\widetilde{D}} \subseteq \omega_D^0$ can be seen as $(\nu_D)_*\omega_{\widetilde{D}}^0 \hookrightarrow \omega_D^0$. He showed that $(\nu_X)_*\omega_{\widetilde{X}}^0 = \omega_X^0$ is equivalent to X being normal crossing in codimension one for reduced equidimensional spaces X which are free in codimension one. Here freeness means Gorenstein with Cohen–Macaulay ω-Jacobian ideal. As the latter coincides with the Jacobian ideal for complete intersections (see [Pie79, Prop. 1]), this generalizes the classical freeness of divisors which holds true in codimension one.

Multi-logarithmic differential forms generalize Saito's logarithmic differential forms replacing hypersurfaces $D \subseteq Y$ by subspaces $X \subseteq Y$ of codimension $k \geq 2$. They were first introduced with meromorphic poles along reduced complete intersections by Aleksandrov and Tsikh (see [AT01, AT08]), later with simple poles by Aleksandrov (see [Ale12, §3]) and recently along reduced Cohen–Macaulay and reduced equidimensional spaces by Aleksandrov (see [Ale14, §10]) and by Pol (see [Pol16, §4.1]). The precise relation of the forms with simple and meromorphic poles was clarified by Pol (see [Pol16, Prop. 3.1.33]). Here we consider only multi-logarithmic forms with simple poles.

The $\mathscr{O}_Y$-modules $\Omega^q(\log X/C)$ of multi-logarithmic q-forms on Y along X depend on the choice of divisors $D_1, \dots, D_k$ defining a reduced complete intersection $C = D_1 \cap \cdots \cap D_k \subseteq Y$ such that $X \subseteq C$. Consider the divisor $D = D_1 \cup \cdots \cup D_k$ defined by $h = h_1 \cdots h_k \in \mathscr{O}_Y$. Due to Aleksandrov and Pol there is a multi-logarithmic residue sequence

$$0 \longrightarrow \Sigma\Omega_Y^q \longrightarrow \Omega^q(\log X/C) \xrightarrow{\operatorname{res}_{X/C}^q} \omega_X^{q-k} \longrightarrow 0 \tag{1.1}$$

where $\Sigma = \mathcal{I}_C(D)$ is obtained from the ideal $\mathcal{I}_C$ of $C \subseteq Y$ and ω_X^p is the $\mathscr{O}_X$-module of regular meromorphic p-forms on X (see [Ale14, §10] and [Pol16, §4.1.3]). Pol introduced an $\mathscr{O}_Y$-module $\mathrm{Der}^k(-\log X/C)$ of logarithmic k-vector fields on Y along X and a kind of Jacobian ideal $\mathcal{J}_{X/C}$ of X that fit into the short exact sequence dual to (1.1) for $q = k$

$$0 \longleftarrow \mathcal{J}_{X/C} \xleftarrow{\langle -,\alpha_X\rangle} \Theta_Y^k \longleftarrow \mathrm{Der}^k(-\log X/C) \longleftarrow 0 \tag{1.2}$$

where $\Theta_Y^q = \bigwedge_{\mathscr{O}_Y}^q \Theta_Y$ and $\begin{bmatrix} \alpha_X \\ h_1,\dots,h_k \end{bmatrix} \in \omega_X^0$ is a fundamental form of X (see [Pol16, §4.2.2-3]). Notably the duality applied here is $-^\Sigma = \mathrm{Hom}_{\mathscr{O}_Y}(-,\Sigma)$. Pol showed that Cohen–Macaulayness of $\mathcal{J}_{X/C}$ serves as a further generalization of freeness. In fact the property is independent of C (see [Pol16, Prop. 4.2.21]) and $\mathcal{J}_{X/C}$ coincides with the ω-Jacobian ideal in case X is Gorenstein (see [Pol16, §4.2.5]). By relating Σ- and $\mathscr{O}_Y$-duality Pol established the following major result (see [Pol16, Thm. 4.2.22] or [Pol15]). In particular it generalizes Saito's original definition of freeness to the case $k > 1$.

Theorem 1.1 (Pol). *Let $X \subseteq C \subseteq Y \cong (\mathbb{C}^n, 0)$ where X is a reduced Cohen–Macaulay germ and C a complete intersection germ, both of codimension $k \geq 1$ in Y. Then*

$$\mathrm{pdim}(\Omega^k(\log X/C)) \geq k - 1$$

with equality equivalent to freeness of X.

In §2 we pursue the main objective of this article: a translation of Theorem 1.1 in terms of general commutative algebra. The role of $\mathscr{O}_Y \twoheadrightarrow \mathscr{O}_C = \mathscr{O}_Y/\mathcal{I}_C$ is played by a map of Gorenstein rings $R \to \overline{R} = R/I$ of codimension $k \geq 2$. For dualizing we use

$$-^I = \mathrm{Hom}_R(-, I), \quad -^\vee = \mathrm{Hom}_R(-, \omega_R), \quad -^{\overline{\vee}} = \mathrm{Hom}_{\overline{R}}(-, \overline{\omega}_R)$$

where ω_R is a canonical module for R and $\overline{\omega}_R = \overline{R} \otimes_R \omega_R$, which is a canonical module for $\overline{R}$ due to the Gorenstein hypothesis (see Notation 2.1). Modelled after the multi-logarithmic residue sequence (1.1) along $X = C$ we define an I-free approximation of a finitely generated R-module M as a short exact sequence

$$0 \longrightarrow IF \xrightarrow{\iota} M \longrightarrow W \longrightarrow 0$$

where F is free and W is an $\overline{R}$-module. More precisely M plays the role of $\Omega^q(\log X/C)(-D)$ which, as opposed to $\Omega^q(\log X/C)$, is independent of the choice of D. The I-dual sequence

$$0 \longleftarrow V \xleftarrow{\alpha} F^\vee \xleftarrow{\lambda} M^I \longleftarrow 0$$

plays the role of the Σ-dual sequence (1.2) for $X = C$. In Proposition 2.13 we show that M is I-reflexive if and only if W is the $\overline{R}$-dual of V. Our main result is

Theorem 1.2. *Let R be a Gorenstein local ring and let I be an ideal of R of height $k \geq 2$ such that $\overline{R} = R/I$ is Gorenstein. Consider an I-free approximation*

$$0 \longrightarrow IF \xrightarrow{\iota} M \xrightarrow{\rho} W \longrightarrow 0$$

of an I-reflexive finitely generated R-module M with $W \neq 0$ and the corresponding I-dual

$$0 \longleftarrow V \xleftarrow{\alpha} F^\vee \xleftarrow{\lambda} M^I \longleftarrow 0.$$

Then $W = V^{\overline{\vee}}$ and V is a maximal Cohen–Macaulay $\overline{R}$-module if and only if $\mathrm{G\text{-}dim}(M) \leq k - 1$. *In this latter case $V = W^{\overline{\vee}}$ is ($\overline{\omega}_R$-)reflexive. Unless $\overline{\alpha} := \overline{R} \otimes \alpha$ is injective,* $\mathrm{G\text{-}dim}(M) \geq k - 1$.

Due to the Gorenstein hypothesis, Theorem 1.2 applies to the complete intersection ring $\overline{R} = \mathscr{O}_C$, but in general not to $\overline{R} = \mathscr{O}_X$. In §2.5 we describe a construction to restrict the support of an I-free approximation to the locus defined by an ideal $J \trianglelefteq R$ with $I \subseteq J$. Lemma 3.15 shows that it is made in a way such that the multi-logarithmic residue sequence along X is obtained from that along C by restricting with $J = \mathcal{I}_X$. Corollary 2.29 extends Theorem 1.2 to this generalized setup.

In §3 we apply our results to multi-logarithmic forms. We define $\mathscr{O}_Y$-submodules $\mathrm{Der}^q(-\log X) \subseteq \Theta_Y^q$ of logarithmic q-vector fields on Y along X independent of C and show that $\mathrm{Der}^k(-\log X) = \mathrm{Der}^k(-\log X/C)$. We further define Jacobian $\mathscr{O}_X$-modules $\mathcal{J}_X^{n-q} \subseteq \mathscr{O}_X \otimes_{\mathscr{O}_Y} \Theta_Y^{q-k}$ of X independent of C and Y such that $\mathcal{J}_X^{\dim X} = \mathcal{J}_{X/C}$. The Σ-dual of the multi-logarithmic residue sequence reads

$$0 \longleftarrow \mathcal{J}_X^{n-q} \xleftarrow{\alpha^X} \Theta_Y^q \longleftarrow \mathrm{Der}^q(-\log X) \longleftarrow 0$$

where α^X is contraction by α_X. As a consequence of Corollary 2.29 we obtain the following result which is due to Pol in case $q = k$ (see [Pol16, Prop. 4.2.17, Thm. 4.2.22]).

Theorem 1.3. *Let $X \subseteq C \subseteq Y \cong (\mathbb{C}^n, 0)$ where X is a reduced Cohen–Macaulay germ and C a complete intersection germ, both of codimension $k \geq 2$ in Y. For $k \leq q < n$, $\omega_X^{q-k} = \mathrm{Hom}_{\mathscr{O}_X}(\mathcal{J}_X^{n-q}, \omega_X)$ where $\omega_X = \mathrm{Hom}_{\mathscr{O}_C}(\mathscr{O}_X, \mathscr{O}_C)(D)$ and $\mathrm{pdim}(\Omega^q(\log X/C)) \geq k-1$. Equality holds if and only if $\mathcal{J}_X^{n-q}$ is maximal Cohen–Macaulay. In this latter case $\mathcal{J}_X^{n-q} = \mathrm{Hom}_{\mathscr{O}_X}(\omega_X^{q-k}, \omega_X)$ is ω_X-reflexive.*

The analogy with the hypersurface case (see [Sai80, (1.8)]) now raises the question whether $\mathcal{J}_X^{n-q}$ being maximal Cohen–Macaulay for $q = k$ implies the same for all $q > k$. An explicit description of the Jacobian modules is given in Remark 3.25.

Acknowledgments. We thank Delphine Pol and the anonymous referee for helpful comments.

2. Residual duality over Gorenstein rings

For this section we fix a Cohen–Macaulay local ring R with $n := \dim(R)$ and an ideal $I \trianglelefteq R$ with $k := \mathrm{height}(I) \geq 2$ defining a Cohen–Macaulay factor ring $\overline{R} := R/I$. These fit into a short exact sequence

$$0 \longrightarrow I \longrightarrow R \xrightarrow{\pi} \overline{R} \longrightarrow 0. \tag{2.1}$$

Note that (see [BH93, Thm. 2.1.2.(b), Cor. 2.1.4])

$$n - \dim(\overline{R}) = \mathrm{grade}(I) = \mathrm{height}(I) = k \geq 2.$$

In particular I is a regular ideal of R and hence any $\overline{R}$-module is R-torsion.

We assume further that R admits a canonical module ω_R. Then also $\overline{R}$ admits a canonical module $\omega_{\overline{R}}$ (see [BH93, Thm. 3.3.7]).

Notation 2.1. Abbreviating $\overline{\omega}_R := \overline{R} \otimes_R \omega_R$ we deal with the following functors

$$-^* := \mathrm{Hom}_R(-, R), \qquad -^\vee := \mathrm{Hom}_R(-, \omega_R),$$
$$-^I := \mathrm{Hom}_R(-, I\omega_R), \quad -^{\overline{\vee}} := \mathrm{Hom}_R(-, \overline{\omega}_R).$$

In general $\overline{\omega}_R \not\cong \omega_{\overline{R}}$ and $-^{\overline{\vee}}$ is not the duality of $\overline{R}$-modules. For an $\overline{R}$-module N, $N^* = \mathrm{Hom}_{\overline{R}}(N, \overline{R})$ but $N^\vee$ means either $\mathrm{Hom}_R(N, \omega_R)$ or $\mathrm{Hom}_{\overline{R}}(N, \omega_{\overline{R}})$, depending on the context. For R-modules M and N, we denote the canonical evaluation map by

$$\delta_{M,N} \colon M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, N), N), \quad m \mapsto (\varphi \mapsto \varphi(m)).$$

Whenever applicable we use an analogous notation for $\overline{R}$-modules. We denote canonical isomorphisms as equalities.

Lemma 2.2. *Let N be an $\overline{R}$-module. Then $\operatorname{Ext}^i_R(N, \omega_R) = 0$ for $i < k$ and $N^I = 0$.*

Proof. The first vanishing is due to Ischebeck's Lemma (see [HK71, Satz 1.9]), the second holds because ω_R and hence $I\omega_R$ is torsion free (see [BH93, Thm. 2.1.2.(c)]) whereas N is torsion. $\square$

2.1. **I-duality and I-free approximation.**

Lemma 2.3. *There is a canonical identification $\omega_R = I^I$ and a canonical inclusion $I \hookrightarrow \omega_R^I$. They combine to the map $\delta_{I,I\omega_R}\colon I \to I^{II}$ which is an isomorphism if R is Gorenstein.*

Proof. Applying $-^\vee$ to (2.1) and $\operatorname{Hom}_R(I, -)$ to $I\omega_R \hookrightarrow \omega_R$ yields an exact sequence with a commutative triangle

(2.2)
$$\begin{tikzcd}
\operatorname{Ext}^1_R(\overline{R}, \omega_R) & I^\vee \arrow[l] & \omega_R \arrow[l] \arrow[dl, "\mu"] & \overline{R}^\vee \arrow[l] & 0 \arrow[l] \\
& I^I. \arrow[u, hook] & & &
\end{tikzcd}$$

The diagonal map sends $\varepsilon \in \omega_R$ to the multiplication map $\mu(\varepsilon)\colon I \to I\omega_R$, $x \mapsto x \cdot \varepsilon$. With Lemma 2.2 it follows that $\omega_R = I^\vee = I^I$.

There is an isomorphism $R \cong \operatorname{End}_R(\omega_R)$ sending each element to the corresponding multiplication map (see [BH93, Thm. 3.3.4.(d))]). Applying $\operatorname{Hom}_R(\omega_R, -)$ to $I\omega_R \hookrightarrow \omega_R$ yields a commutative square

(2.3)
$$\begin{tikzcd}
R \arrow[r, "\cong"'] & \operatorname{End}_R(\omega_R) \\
I \arrow[u, hook] \arrow[r, "\delta'"] & \omega_R^I. \arrow[u, hook]
\end{tikzcd}$$

If R is Gorenstein, then $\omega_R^I = \operatorname{Hom}_R(R, I) = I$ and δ' is an isomorphism.

Combined with the above identification $\omega_R = I^I$, δ' defines a map $\delta\colon I \to I^{II}$. Since

$$\delta(x)(\mu(\varepsilon)) = \delta'(x)(\varepsilon) = x \cdot \varepsilon = \mu(\varepsilon)(x) = \delta_{I,I\omega_R}(x)(\mu(\varepsilon))$$

for all $x \in I$ and $\varepsilon \in \omega_R$, in fact $\delta = \delta_{I,I\omega_R}$. $\square$

Definition 2.4. If F is a free R-module, then we call $IF = I \otimes_R F$ an *I-free module.* An R-module M is called *I-reflexive* if $\delta_{M,I\omega_R}\colon M \to M^{II}$ is an isomorphism.

Proposition 2.5. *Let F be a free R-module F. Then $F^\vee = (IF)^I$ by restriction. The adjunction map $IF \to F^{\vee I}$ is induced by the isomorphism δ_{F,ω_R} and identifies with $\delta_{IF,I\omega_R}$. In case R is Gorenstein, IF is I-reflexive.*

Proof. Applying $\operatorname{Hom}_R(F, -)$ to μ in (2.2) yields $F^\vee = (IF)^I$ by Hom-tensor adjunction.

Applying $F \otimes_R -$ to (2.3) yields a commutative square

$$\begin{tikzcd}
F \arrow[r, "\delta_{F,\omega_R}", "\cong"'] & F^{\vee\vee} \\
IF \arrow[u, hook] \arrow[r] & F^{\vee I} \arrow[u, hook]
\end{tikzcd}$$

where the bottom row is adjunction. In fact, using Lemma 2.3,

$$\begin{aligned}
IF = I \otimes_R F \to F \otimes_R \omega_R^I &= F \otimes_R \operatorname{Hom}_R(\omega_R, I\omega_R) \\
&= \operatorname{Hom}_R(F \otimes_R \omega_R, I\omega_R) \\
&= \operatorname{Hom}_R(F \otimes_R \operatorname{Hom}_R(R, \omega_R), I\omega_R) \\
&= \operatorname{Hom}_R(\operatorname{Hom}_R(F \otimes_R R, \omega_R), I\omega_R) \\
&= \operatorname{Hom}_R(\operatorname{Hom}_R(F, \omega_R), I\omega_R) = F^{\vee I}, \\
x \cdot e \mapsto (\psi &\mapsto x \cdot \psi(e)).
\end{aligned}$$

Identifying $F^\vee = (IF)^I$ using Lemma 2.3 yields with the map μ in diagram (2.2)

$$\varepsilon = \psi(e) \leftrightarrow \mu(\varepsilon) \implies x \cdot \psi(e) = x \cdot \varepsilon = \mu(\varepsilon)(x).$$

Adjunction thus becomes identified with $\delta_{IF, I\omega_R}$. The last claim is due to Lemma 2.3. $\square$

Definition 2.6. Let M be a finitely generated R-module. We call a short exact sequence

$$0 \longrightarrow IF \xrightarrow{\iota} M \xrightarrow{\rho} W \longrightarrow 0 \tag{2.4}$$

where F is free and $IW = 0$ an *I-free approximation of M* with *support* $\operatorname{Supp}(W)$. We consider W as an $\overline{R}$-module. The inclusion map $\iota \colon IF \hookrightarrow F = M$ defines the *trivial I-free approximation*

$$0 \longrightarrow IF \longrightarrow F \longrightarrow F/IF \longrightarrow 0.$$

A *morphism of I-free approximations* is a morphism of short exact sequences.

Lemma 2.7. *For any I-free approximation (2.4), ι fits into a unique commutative triangle*

$$\begin{tikzcd} & F \\ IF \arrow[ur, hook] \arrow[r, hook, "\iota"] & M. \arrow[u, "\kappa"'] \end{tikzcd} \tag{2.5}$$

If ι^{-1} denotes the choice of any preimage under ι, then $\kappa(m) = \iota^{-1}(xm)/x$ for any $x \in I \cap R^{\mathrm{reg}}$. If M is maximal Cohen–Macaulay, then κ is surjective. In particular, (2.4) becomes trivial if in addition κ injective.

Proof. Applying $\operatorname{Hom}_R(-, F)$ to (2.4) yields

$$\operatorname{Ext}^1_R(W, F) \longleftarrow \operatorname{Hom}_R(IF, F) \xleftarrow{\iota^*} \operatorname{Hom}_R(M, F) \longleftarrow \operatorname{Hom}_R(W, F) \longleftarrow 0.$$

By Ischebeck's Lemma (see [HK71, Satz 1.9]), $\operatorname{Ext}^1_R(W, F) = 0 = \operatorname{Hom}_R(W, F)$ making ι^* an isomorphism. Then κ is the preimage of the canonical inclusion $IF \hookrightarrow F$ under ι^*. The formula for κ follows immediately.

Since $\operatorname{coker}(\kappa)$ is a homomorphic image of F/IF, $\dim(\operatorname{coker}(\kappa)) \leq n - k \leq n - 2$. If M is maximal Cohen–Macaulay, then $\operatorname{depth}(\operatorname{coker}(\kappa)) \geq n - 1$ by the Depth Lemma (see [BH93, Prop. 1.2.9]). This forces $\operatorname{coker}(\kappa) = 0$ (see [BH93, Prop. 1.2.13]) and makes κ surjective. $\square$

By functoriality of the cokernel, any $\varphi \in F^\vee$ gives rise to a commutative diagram

(2.6)
$$\begin{tikzcd}
0 \arrow[r] & I\omega_R \arrow[r] & \omega_R \arrow[r, "\pi_\omega"] & \overline{\omega}_R \arrow[r] & 0 \\
 & & F \arrow[u, "\varphi"'] & & \\
0 \arrow[r] & IF \arrow[r, "\iota"] \arrow[uu, "\varphi|_{IF}"'] \arrow[ur, hook] & M \arrow[r, "\rho"] \arrow[u, "\kappa"'] & W \arrow[r] \arrow[uu, dashed, "\overline{\varphi}"'] & 0
\end{tikzcd}$$

with top exact row induced by (2.1) and bottom row (2.4). This defines a map

(2.7)
$$\begin{aligned} W^{\overline{\vee}} &\longleftarrow F^\vee \\ \overline{\varphi} &\longleftarrow\!\shortmid \varphi. \end{aligned}$$

Applying $\mathrm{Hom}_R(F, -)$ to the upper row of (2.6) yields a short exact sequence

(2.8)
$$0 \longrightarrow F^I \longrightarrow F^\vee \longrightarrow F^{\overline{\vee}} \longrightarrow 0.$$

By Lemma 2.2 applying $-^I$ to (2.4) and (2.5) yields the exact diagonal sequence and the triangle of inclusions with vertex F^I in the following commutative diagram.

(2.9)
$$\begin{tikzcd}
0 & V \arrow[l] \arrow[d, equal] & F^{\overline{\vee}} \arrow[l, "\overline{\alpha}"'] & M^I/F^I \arrow[l, "\overline{\lambda}"'] & 0 \arrow[l] \arrow[dl] \\
0 & V \arrow[l] \arrow[dd, hook] & F^\vee \arrow[l, "\alpha"'] \arrow[u, two heads] \arrow[d, equal] & M^I \arrow[l, "\lambda"'] \arrow[u, two heads] \arrow[dl, "\iota^I"'] & 0 \arrow[l] \\
 & & (IF)^I \arrow[dl] & F^I \arrow[l, hook'] \arrow[u, hook, "\kappa^I"'] & \\
 & \mathrm{Ext}^1_R(W, I\omega_R) & & &
\end{tikzcd}$$

By Proposition 2.5, the identification $F^\vee = (IF)^I$ in diagram (2.9) is given by

$$\varphi \leftrightarrow \varphi|_{IF} = \varphi \circ \kappa \circ \iota$$

in diagram (2.6). It defines the map λ with cokernel α. For $\psi \in M^I$, $\lambda(\psi)$ is defined by

$$\lambda(\psi)|_{IF} = \psi \circ \iota.$$

With $\mathrm{Ext}^1_R(W, I\omega_R)$ also V is an $\overline{R}$-module. Using (2.8) the Snake Lemma yields the short exact upper row of (2.9). By Lemma 2.2 the commutative square $\mathrm{Hom}_R(IF \hookrightarrow M, I\omega_R \hookrightarrow \omega_R)$ reads

$$\begin{tikzcd}
(IF)^I \arrow[d, hook] & M^I \arrow[l, "\iota^I"'] \arrow[d, hook] \\
(IF)^\vee & M^\vee. \arrow[l, "\iota^\vee"', "\cong"]
\end{tikzcd}$$

This allows one to check equalities of maps $M \to \omega_R$ after precomposing with ι. It follows that

(2.10)
$$\varphi \circ \kappa \in M^I \iff \varphi \in \lambda(M^I) \implies \varphi = \lambda(\varphi \circ \kappa)$$

for any $\varphi \in F^\vee$.

Definition 2.8. We call the middle row

$$0 \longleftarrow V \xleftarrow{\alpha} F^\vee \xleftarrow{\lambda} M^I \longleftarrow 0 \tag{2.11}$$

of diagram (2.9) the *I-dual* of the I-free approximation (2.4). We set

$$W' := \operatorname{Ext}^1_R(V, I\omega_R). \tag{2.12}$$

Lemma 2.9. *For any I-free approximation (2.4) the map (2.7) factors through the map α in (2.9) defining an inclusion $\nu\colon V \to W^\triangledown$, that is,*

$$\begin{tikzcd}
W^\triangledown & V \arrow[l, hook', "\nu"'] & F^\vee, \arrow[l, two heads, "\alpha"'] \\
\overline{\varphi} & & \varphi. \arrow[ll, mapsto]
\end{tikzcd}$$

Proof. By diagrams (2.6) and (2.9), equivalence (2.10) and exactness properties of Hom,

$$\overline{\varphi} = 0 \iff \overline{\varphi} \circ \rho = 0 \iff \varphi \circ \kappa \in M^I \iff \varphi \in \lambda(M^I) \iff \alpha(\varphi) = 0. \qquad \square$$

Remark 2.10. By Lemma 2.2 applying $\operatorname{Hom}_R(W, -)$ to the upper row of diagram (2.6) yields

$$W^\triangledown = \operatorname{coker} \operatorname{Hom}_R(W, \pi_\omega) \cong \operatorname{Ext}^1_R(W, I\omega_R).$$

The inclusion of V in the latter in diagram (2.9) uses $\operatorname{coker} \iota^I \hookrightarrow \operatorname{Ext}^1_R(W, I\omega_R)$. The relation with the inclusion ν in Lemma 2.9 is clarified by the double complex obtained by applying $\operatorname{Hom}_R(-,-)$ to (2.4) and the upper row of (2.6). By Lemma 2.2 it expands to a commutative diagram with exact rows and columns

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
\operatorname{Ext}^1_R(W, I\omega_R) \arrow[d] & (IF)^I \arrow[l] \arrow[d] & M^I \arrow[l, "\iota^I"'] \arrow[d] & 0 \arrow[l] \arrow[d] & \\
0 & (IF)^\vee \arrow[l] \arrow[d] & M^\vee \arrow[l, "\iota^\vee"'] \arrow[d] & 0 \arrow[l] \arrow[d, "\operatorname{Hom}_R(W,\pi_\omega)"] & \\
& (IF)^\triangledown & M^\triangledown \arrow[l] & W^\triangledown \arrow[l] \arrow[d, "\cong"'] & 0 \arrow[l] \\
& & & \operatorname{Ext}^1_R(W, I\omega_R). &
\end{tikzcd}$$

An element $\alpha(\varphi) \in V$ with $\varphi \in F^\vee$ maps to $\varphi|_{IF} \in (IF)^I$, to $\varphi \circ \kappa \in M^\vee$ and to $\overline{\varphi} \in W^\triangledown$.

2.2. ***I*-reflexivity over Gorenstein rings.** In this subsection we assume that R is Gorenstein and study I-reflexivity of modules M in terms of an I-free approximation (2.4). With the Gorenstein hypothesis $F^\vee$ is free and hence

$$\operatorname{Ext}^1_R(F^\vee, -) = 0. \tag{2.13}$$

Proposition 2.11. *Assume that R is Gorenstein. For any I-free approximation (2.4) and W' as in (2.12) there is a commutative square*

$$\begin{tikzcd}
M \arrow[r, two heads, "\rho"] \arrow[d, "\delta_{M,I\omega_R}"] & W \arrow[d, "\overline{\delta}"] \\
M^{II} \arrow[r, two heads, "\rho'"] & W'
\end{tikzcd}$$

and $\overline{\delta}$ is an isomorphism if and only if M is I-reflexive.

Proof. Consider the following commutative diagram whose rows are (2.4) and obtained by applying $-^I$ to the triangle with vertex $F^\vee$ in diagram (2.9).

(2.14)
$$
\begin{tikzcd}
 & & F \arrow[dd, bend left=90, "\delta_{F,\omega_R}", "\cong"'] & & \\
0 \arrow[r] & IF \arrow[ur, hook] \arrow[r, "\iota"] \arrow[d, "\cong"', "\delta_{IF,I\omega_R}"] & M \arrow[u, "\kappa"'] \arrow[r, "\rho"] \arrow[d, "\delta_{M,I\omega_R}"] & W \arrow[r] \arrow[d, dashed, "\overline{\delta}"] & 0 \\
0 \arrow[r] & (IF)^{II} \arrow[r, "\iota^{II}"] \arrow[d, equal] & M^{II} \arrow[r, "\rho'"] & W' \arrow[r] & 0 \\
 & F^{\vee I} \arrow[ur, "\lambda^I"] \arrow[r, hook] & F^{\vee\vee}. & &
\end{tikzcd}
$$

The latter is a short exact sequence by Lemma 2.2 and (2.13). The commutative squares in diagram (2.14) are due to functoriality of δ and the cokernel. The claimed equivalence then follows from the Snake Lemma. Proposition 2.5 yields the part of diagram (2.14) involving δ_{F,ω_R}. This part is just added for clarification but not needed for the proof. $\square$

Lemma 2.12. *Assume that R is Gorenstein and consider an I-free approximation (2.4). Then the maps ν from Lemma 2.9 and $\overline{\delta}$ from Proposition 2.11 fit into a commutative square*

$$
\begin{tikzcd}
W \arrow[r, "\delta_{W,\overline{\omega}_R}"] \arrow[d, "\overline{\delta}"] & W^{\overline{\vee\vee}} \arrow[d, "\nu^{\overline{\vee}}"] \\
W' & V^{\overline{\vee}}. \arrow[l, "\xi", "\cong"']
\end{tikzcd}
$$

Proof. Consider the double complex obtained by applying $\mathrm{Hom}_R(-,-)$ to the middle and top rows of diagrams (2.9) and (2.6). By Lemma 2.2 and (2.13) it expands to a commutative diagram with exact rows and columns

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] \arrow[d] & F^{\vee I} \arrow[r, "\lambda^I"] \arrow[d] & M^{II} \arrow[r, "\rho'"] \arrow[d] & W' \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] \arrow[d] & F^{\vee\vee} \arrow[r, "\lambda^\vee"] \arrow[d, "(\pi_\omega)_*"] & M^{I\vee} \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & V^{\overline{\vee}} \arrow[r, "\alpha^{\overline{\vee}}"] & F^{\vee\overline{\vee}} \arrow[r, "\lambda^{\overline{\vee}}"] \arrow[d] & M^{I\overline{\vee}} & & \\
 & & 0. & & &
\end{tikzcd}
$$

The Snake Lemma yields an isomorphism $\xi\colon V^{\triangledown} \to W'$. Attaching the square of Proposition 2.11, the relation $\overline{\delta}(w) = \xi(\widetilde{\psi})$ is given by the diagram chase

$$
\begin{tikzcd}
 & & m \arrow[r, mapsto] \arrow[d, mapsto] & w \arrow[d, mapsto] \\
 & & \delta_{M,I\omega_R}(m) \arrow[r, mapsto] \arrow[d, mapsto] & \overline{\delta}(w) \\
 & \psi \arrow[r, mapsto] \arrow[d, mapsto] & \psi \circ \lambda = \delta_{M,I\omega_R}(m) & \\
\widetilde{\psi} \arrow[r, mapsto] & \widetilde{\psi} \circ \alpha = \pi_\omega \circ \psi. & &
\end{tikzcd}
$$

Using implication (2.10), diagram (2.6) and Lemma 2.9, one deduces that, with $x \in I \cap R^{\mathrm{reg}}$ and $v = \alpha(\varphi)$,

$$
\begin{aligned}
x\varphi \circ \kappa \in M^I \implies & x\varphi = \lambda(x\varphi \circ \kappa) \\
\implies & x\psi(\varphi) = \psi(x\varphi) = (\psi \circ \lambda)(x\varphi \circ \kappa) = \delta_{M,I\omega_R}(m)(x\varphi \circ \kappa) = x(\varphi \circ \kappa)(m) \\
\implies & \psi(\varphi) = (\varphi \circ \kappa)(m) \\
\implies & \widetilde{\psi}(v) = (\widetilde{\psi} \circ \alpha)(\varphi) = (\pi_\omega \circ \psi)(\varphi) = (\pi_\omega \circ \varphi \circ \kappa)(m) = \overline{\varphi}(w) \\
& = (\nu \circ \alpha)(\varphi)(w) = \nu(\alpha(\varphi))(w) = \nu(v)(w) \\
& = \delta_{W,\varpi_R}(w)(\nu(v)) = \nu^{\triangledown}(\delta_{W,\varpi_R}(w))(v) = (\nu^{\triangledown} \circ \delta_{W,\varpi_R})(w)(v) \\
\implies & \widetilde{\psi} = (\nu^{\triangledown} \circ \delta_{W,\varpi_R})(w) \\
\implies & \overline{\delta}(w) = \xi(\widetilde{\psi}) = (\xi \circ \nu^{\triangledown} \circ \delta_{W,\varpi_R})(w) \\
\implies & \overline{\delta} = \xi \circ \nu^{\triangledown} \circ \delta_{W,\varpi_R}. \qquad \square
\end{aligned}
$$

Proposition 2.13. *Assume that R is Gorenstein and consider an I-free approximation (2.4). Then M is I-reflexive if and only if the map $\nu^{\triangledown} \circ \delta_{W,\varpi_R}$ with ν from Lemma 2.9 identifies $W = V^{\triangledown}$.*

Proof. The claim follows from Proposition 2.11 and Lemma 2.12. $\square$

Lemma 2.14. *Assume that R is Gorenstein and consider an I-free approximation (2.4). Then the map ν from Lemma 2.9 fits into a commutative diagram*

$$
\begin{tikzcd}
W^{\triangledown} & V \arrow[l, "\nu"'] \arrow[d, "\delta_{V,\varpi_R}"] \\
W^{\triangledown\triangledown\triangledown} \arrow[u, "\delta^{\triangledown}_{W,\varpi_R}"] & V^{\triangledown\triangledown}. \arrow[l, "\nu^{\triangledown\triangledown}"] \arrow[ul, "(\nu^{\triangledown} \circ \delta_{W,\varpi_R})^{\triangledown}"']
\end{tikzcd}
$$

Proof. For any $v \in V$ and $w \in W$ we have

$$
\begin{aligned}
(\delta^{\triangledown}_{W,\varpi_R} \circ \nu^{\triangledown\triangledown} \circ \delta_{V,\varpi_R})(v)(w) &= \delta^{\triangledown}_{W,\varpi_R}(\nu^{\triangledown\triangledown}(\delta_{V,\varpi_R}(v)))(w) = \delta^{\triangledown}_{W,\varpi_R}(\delta_{V,\varpi_R}(v) \circ \nu^{\triangledown})(w) \\
&= (\delta_{V,\varpi_R}(v) \circ \nu^{\triangledown})(\delta_{W,\varpi_R}(w)) = \delta_{V,\varpi_R}(v)(\delta_{W,\varpi_R}(w) \circ \nu) \\
&= \delta_{W,\varpi_R}(w)(\nu(v)) = \nu(v)(w)
\end{aligned}
$$

and hence $\nu = \delta^{\triangledown}_{W,\varpi_R} \circ \nu^{\triangledown\triangledown} \circ \delta_{V,\varpi_R}$ as claimed. $\square$

Corollary 2.15. *Assume that R is Gorenstein and consider an I-free approximation* (2.4) *of an I-reflexive R-module M. Then V in diagram* (2.9) *is ($\overline{\omega}_R$-)reflexive if and only if ν in Lemma 2.9 identifies $V = W^{\triangledown}$.*

Proof. The claim follows from Proposition 2.13 and Lemma 2.14. □

2.3. **R-dual I-free approximation.** In this subsection we consider the R-dual of an I-free approximation (2.4). The interesting part of the long exact Ext-sequence of $-^{\vee}$ applied to (2.4) turns out to be

$$0 \leftarrow \operatorname{Ext}^k_R(M,\omega_R) \leftarrow \operatorname{Ext}^k_R(W,\omega_R) \xleftarrow{\beta} \operatorname{Ext}^{k-1}_R(IF,\omega_R) \leftarrow \operatorname{Ext}^{k-1}_R(M,\omega_R) \leftarrow 0. \tag{2.15}$$

In fact, applying $-^{\vee}$ to (2.1) yields (see Lemma 2.17 and [BH93, Thm. 3.3.10.(c).(ii)])

$$\operatorname{Ext}^i_R(IF,\omega_R) = F^* \otimes_R \operatorname{Ext}^i_R(I,\omega_R) = F^* \otimes_R \operatorname{Ext}^{i+1}_R(\overline{R},\omega_R) = 0 \text{ for } i \neq 0, k-1.$$

In case both R and $\overline{R}$ are Gorenstein, we will identify the map β to its image with the map $\overline{\alpha}$ in (2.9) (see Corollary 2.21). In §2.4 this fact will serve to relate the Gorenstein dimension of M to the depth of V.

In order to describe the map β in (2.15) we fix a canonical module ω_R of R with an injective resolution $(E^\bullet, \partial^\bullet)$,

$$0 \longrightarrow \omega_R \longrightarrow E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} E^2 \xrightarrow{\partial^2} \cdots.$$

We use it to fix representatives

$$\operatorname{Ext}^i_R(-,\omega_R) := H^i \operatorname{Hom}_R(-,E^\bullet).$$

Then (see [BH93, Thms. 3.3.7.(b), 3.3.10.(c).(ii)])

$$H^i \operatorname{Ann}_{E^\bullet}(I) = H^i \operatorname{Hom}(\overline{R},E^\bullet) = \operatorname{Ext}^i_R(\overline{R},\omega_R) = \delta_{i,k} \cdot \omega_{\overline{R}} \tag{2.16}$$

where

$$\omega_{\overline{R}} := H^k \operatorname{Ann}_{E^\bullet}(I)$$

is a canonical module of $\overline{R}$.

In the sequel we explicit the maps of the following commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}^k_R(W,\omega_R) \arrow[ddd, "\gamma"', "\cong"] & & \operatorname{Ext}^{k-1}_R(IF,\omega_R) \arrow[ll, "\beta"'] \\
& & F^* \otimes_R \operatorname{Ext}^{k-1}_R(I,\omega_R) \arrow[u, "\chi", "\cong"'] \\
& & F^* \otimes_R H^{k-1}(E^\bullet/\operatorname{Ann}_{E^\bullet}(I)) \arrow[u, "F^*\otimes H^{k-1}(\tau^\bullet)", "\cong"'] \arrow[d, "F^*\otimes\zeta", "\cong"'] \\
\operatorname{Hom}_{\overline{R}}(W,\omega_{\overline{R}}) = W^\vee & V' \arrow[l, hook', "\nu'"'] & F^* \otimes_R \omega_{\overline{R}} \arrow[l, two heads, "\alpha'"']
\end{tikzcd} \tag{2.17}$$

which defines the map $\nu' \circ \alpha'$ and its image V'. The maps $\tau^\bullet$, χ, ζ, γ and α' are described in Lemmas 2.16, 2.17, 2.18, 2.19 and Proposition 2.20 respectively.

Lemma 2.16. *For any injective R-module E there is a canonical isomorphism*

$$\tau \colon E/\operatorname{Ann}_E(I) \to \operatorname{Hom}_R(I,E), \quad \overline{e} \mapsto - \cdot e = (x \mapsto x \cdot e).$$

In particular, there is a canonical isomorphism $\tau^\bullet \colon E^\bullet/\operatorname{Ann}_{E^\bullet}(I) \to \operatorname{Hom}_R(I,E^\bullet)$.

Proof. Applying the exact functor $\operatorname{Hom}_R(-,E)$ to (2.1) yields a short exact sequence

$$0 \leftarrow \operatorname{Hom}_R(I,E) \leftarrow \operatorname{Hom}_R(R,E) \leftarrow \operatorname{Hom}_R(\overline{R},E) \leftarrow 0.$$

Identifying $E = \operatorname{Hom}_R(R,E)$, $e \mapsto - \cdot e$, and hence

$$\operatorname{Hom}_R(\overline{R},E) = \operatorname{Ann}_E(I) \tag{2.18}$$

yields the claim. $\square$

Lemma 2.17. *For any $i \in \mathbb{N}$ there is a canonical isomorphism*

$$\begin{aligned}\chi_i \colon F^* \otimes_R \operatorname{Ext}^i_R(I,\omega_R) = F^* \otimes_R H^i \operatorname{Hom}_R(I,E^\bullet) \to H^i \operatorname{Hom}_R(IF,E^\bullet) = \operatorname{Ext}^i_R(IF,\omega_R),\\ \varphi \otimes [\psi] \mapsto [\varphi|_{IF} \cdot \widetilde{\psi}(1)] = [(\kappa \circ \iota)^*(\varphi) \cdot \widetilde{\psi}(1)]\end{aligned}$$

where $\widetilde{\psi} \in \operatorname{Hom}_R(R,E^\bullet)$ extends $\psi \in \operatorname{Hom}_R(I,E^\bullet)$. We set $\chi := \chi_{k-1}$.

Proof. For any $i \in \mathbb{N}$ there is a sequence of canonical isomorphisms

$$\begin{aligned}F^* \otimes_R H^i \operatorname{Hom}_R(I,E^\bullet) &= \operatorname{Hom}_R(F, H^i \operatorname{Hom}_R(I,E^\bullet))\\ &= H^i \operatorname{Hom}_R(F, \operatorname{Hom}_R(I,E^\bullet))\\ &= H^i \operatorname{Hom}_R(IF,E^\bullet),\end{aligned}$$

the latter one being Hom-tensor adjunction, sending

$$\begin{aligned}\varphi \otimes [\psi] &\mapsto (f \mapsto \varphi(f) \cdot [\psi] = [\varphi(f) \cdot \psi])\\ &\mapsto [f \mapsto \varphi(f) \cdot \psi]\\ &\mapsto [x \cdot f \mapsto \varphi(f) \cdot \psi(x) = \varphi(x \cdot f) \cdot \widetilde{\psi}(1)] = [\varphi|_{IF} \cdot \widetilde{\psi}(1)]\end{aligned}$$

where $x \in I$ and $f \in F$. $\square$

Lemma 2.18. *There is a connecting isomorphism*

$$\begin{aligned}\zeta \colon H^{k-1}(E^\bullet / \operatorname{Ann}_{E^\bullet}(I)) \to H^k \operatorname{Ann}_{E^\bullet}(I) = \omega_{\overline{R}},\\ [\overline{e}] \mapsto [\partial^{k-1}(e)].\end{aligned}$$

Proof. The connecting homomorphism ζ in degree k of the short exact sequence

$$0 \to \operatorname{Ann}_{E^\bullet}(I) \to E^\bullet \to E^\bullet / \operatorname{Ann}_{E^\bullet}(I) \to 0$$

is an isomorphism since $E^\bullet$ is a resolution and hence $H^i(E^\bullet) = 0$ for $i \geq k-1 \geq 1$. $\square$

Lemma 2.19. *For any $\overline{R}$-module N there is a canonical isomorphism*

$$\begin{aligned}\gamma \colon H^k \operatorname{Hom}_R(N,E^\bullet) \to \operatorname{Hom}_{\overline{R}}(N, H^k \operatorname{Ann}_{E^\bullet}(I)) = N^\vee,\\ [\phi] \mapsto (n \mapsto [\phi(n)]).\end{aligned}$$

Proof. Fix an $\overline{R}$-projective resolution $(P_\star, \delta_\star)$ of N and consider the double complex

$$A^{\star,\bullet} := \operatorname{Hom}_R(P_\star, E^\bullet) = \operatorname{Hom}_{\overline{R}}(P_\star, \operatorname{Hom}_R(\overline{R},E^\bullet)) = \operatorname{Hom}_{\overline{R}}(P_\star, \operatorname{Ann}_{E^\bullet}(I))$$

whose alternate representation is due to Hom-tensor adjunction and (2.18). It yields two spectral sequences with the same limit. By exactness of $\operatorname{Hom}_{\overline{R}}(P_\star,-)$ and (2.16) and using the alternate representation the E_2-page of the first spectral sequence identifies with

$${}'E_2^{p,q} = H^p(H^{\star,q}(A^{\star,\bullet})) = H^p \operatorname{Hom}_{\overline{R}}(P_\star, H^q \operatorname{Ann}_{E^\bullet}(I)) = \delta_{k,q} \cdot H^p \operatorname{Hom}_{\overline{R}}(P_\star, \omega_{\overline{R}}).$$

By exactness of $\operatorname{Hom}_R(-,E^\bullet)$ the E_2-page of the second spectral sequence reads

$${}''E_2^{p,q} = H^q(H^{p,\bullet}(A^{\star,\bullet})) = H^q \operatorname{Hom}_R(H^p P_\star, E^\bullet) = \delta_{p,0} \cdot H^q \operatorname{Hom}_R(N,E^\bullet).$$

So both spectral sequences degenerate. The resulting isomorphism ${}''E_2^{0,k} \to {}'E_2^{0,k}$ is γ. $\square$

Proposition 2.20. *Assume that R is Gorenstein and consider an I-free approximation (2.4). Then the map α' in diagram (2.17) is induced by*

$$\nu' \circ \alpha' \colon F^* \otimes_R \omega_{\overline{R}} = F^* \otimes_R H^k \operatorname{Ann}_{E^\bullet}(I) \to \operatorname{Hom}_{\overline{R}}(W, H^k \operatorname{Ann}_{E^\bullet}(I)) = W^\vee,$$
$$\varphi \otimes [a] \mapsto \overline{\varphi} \cdot [a],$$

where $\varphi \mapsto \overline{\varphi}$ is (2.7) with $\omega_R = R$. In particular, $\operatorname{Ext}_R^k(M, R) = 0$ if ν' is surjective.

Proof. The proof is done by chasing diagram (2.17) and the diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(W, E^{k-1}) \arrow[r, "\rho^*"] \arrow[d, "(\partial^{k-1})_*"] & \operatorname{Hom}_R(M, E^{k-1}) \arrow[r, "\iota^*"] \arrow[d, "(\partial^{k-1})_*"] & \operatorname{Hom}_R(IF, E^{k-1}) \arrow[r] \arrow[d, "(\partial^{k-1})_*"] & 0 \\
0 \arrow[r] & \operatorname{Hom}_R(W, E^k) \arrow[r, "\rho^*"] & \operatorname{Hom}_R(M, E^k) \arrow[r, "\iota^*"] & \operatorname{Hom}_R(IF, E^k) \arrow[r] & 0.
\end{tikzcd}$$

This latter defines the connecting homomorphism β in (2.15) on representatives as $(\rho^*)^{-1} \circ (\partial^{k-1})_* \circ (\iota^*)^{-1}$ where $(\iota^*)^{-1}$ denotes the choice of any preimage under ι^*.

Let $\varphi \otimes [\overline{e}] \in F^* \otimes_R H^{k-1}(E^\bullet / \operatorname{Ann}_{E^\bullet}(I))$. Then by Lemmas 2.16, 2.17, 2.18 and 2.19, and diagram (2.6) with $\omega_R = R$

$$\begin{tikzcd}
 & {[\kappa^*(\varphi) \cdot e]} \arrow[r, mapsto, "H^{k-1}(\iota^*)"] \arrow[d, mapsto] & {[(\iota^* \circ \kappa^*)(\varphi) \cdot e]} \\
{[((\rho^{-1})^* \circ \kappa^*)(\varphi) \cdot \partial^{k-1}(e)]} \arrow[r, mapsto, "H^k(\rho^*)"] \arrow[dd, mapsto, "\gamma"] & {[\kappa^*(\varphi) \cdot \partial^{k-1}(e)]} & \varphi \otimes [- \cdot e] \arrow[u, mapsto, "\chi"] \\
 & & \varphi \otimes [\overline{e}] \arrow[u, mapsto, "F^* \otimes H^{k-1}(\tau^\bullet)"] \arrow[d, mapsto, "F^* \otimes \zeta"] \\
(\pi \circ \varphi \circ \kappa \circ \rho^{-1}) \cdot [\partial^{k-1}(e)] = \overline{\varphi} \cdot [\partial^{k-1}(e)] & & \varphi \otimes [\partial^{k-1}(e)] \arrow[ll, mapsto, "\nu' \circ \alpha'"]
\end{tikzcd}$$

where ρ^{-1} denotes the choice of any preimage under ρ. By diagram (2.6) and Lemma 2.18 the ambiguity of this choice is cancelled when multiplying $(\rho^{-1})^* \circ \kappa^*(\varphi) = \varphi \circ \kappa \circ \rho^{-1}$ with $\partial^{k-1}(e) \in \operatorname{Ann}_{E^\bullet}(I)$.

The particular claim follows from diagram (2.17) and the exact sequence (2.15). $\square$

Corollary 2.21. *Assume that both R and $\overline{R}$ are Gorenstein and consider an I-free approximation (2.4). Then identifying $\overline{\omega}_R = \omega_{\overline{R}}$ (see diagrams (2.9) and (2.17)) makes*

$$\alpha' = \overline{\alpha}, \quad V' = V, \quad \operatorname{Ext}_R^{k-1}(M, R) \cong \ker(\overline{\alpha}) = M^I / F^I.$$

In particular, if M is I-reflexive, then $\operatorname{Ext}_R^k(M, R) = 0$ if and only if V is ($\overline{\omega}_R$-)reflexive.

Proof. Let $\varphi \mapsto \overline{\varphi}$ be (2.7) with $\omega_R = R$. Pick free generators $\varepsilon \in \omega_R$ and $\widetilde{\varepsilon} \in \omega_{\overline{R}}$ inducing the identification $\overline{\omega}_R = \omega_{\overline{R}}$ by sending $\overline{\varepsilon} = \pi_\omega(\varepsilon) \mapsto \widetilde{\varepsilon}$. Then

$$F^\vee \otimes_R \overline{R} = F^* \otimes_R \overline{\omega}_R = F^* \otimes_R \omega_{\overline{R}}, \qquad W^{\overline{\vee}} = W^\vee,$$
$$(\varphi \cdot \varepsilon) \otimes \overline{1} \leftrightarrow \varphi \otimes \overline{\varepsilon} \leftrightarrow \varphi \otimes \widetilde{\varepsilon}, \qquad \overline{\varphi} \cdot \overline{\varepsilon} \leftrightarrow \overline{\varphi} \cdot \widetilde{\varepsilon}.$$

By diagram (2.6) and Lemma 2.9 the map $F^\vee \otimes_R \overline{R} \to W^{\overline{\vee}}$ induced by $\nu \circ \alpha$ sends

$$(\varphi \cdot \varepsilon) \otimes \overline{1} \mapsto \overline{\varphi \cdot \varepsilon} = \pi_\omega \circ ((\varphi \circ \kappa \circ \rho^{-1}) \cdot \varepsilon) = (\pi \circ \varphi \circ \kappa \circ \rho^{-1}) \cdot \pi_\omega(\varepsilon) = \overline{\varphi} \cdot \overline{\varepsilon}.$$

By Proposition 2.20 this map coincides with $\nu' \circ \alpha'$ subject to the above identifications. This shows that $\alpha' = \overline{\alpha}$ and $V' = V$. By the exact sequence (2.15), the commutative diagram (2.17) and the exact upper row of diagram (2.9),

$$\begin{aligned}\operatorname{Ext}_R^{k-1}(M,R) &= \ker(\beta) \cong \ker(\alpha') = \ker(\overline{\alpha}) = M^I/F^I,\\ \operatorname{Ext}_R^{k}(M,R) &= \operatorname{coker}(\beta) \cong \operatorname{coker}(\nu') = W^\vee/\nu'(V').\end{aligned}$$

In particular $\operatorname{Ext}_R^k(M,R) = 0$ if and only if ν' identifies $V' = W^\vee$ or, equivalently, if ν identifies $V = W^{\triangledown}$. The particular claim now follows with Corollary 2.15. $\square$

2.4. **Projective dimension and residual depth.** Assume that R is Gorenstein. Then every finitely generated R-module M has finite Gorenstein dimension $\operatorname{G-dim}(M) < \infty$ (see [Maş00, Thm. 17]). Recall that if M has finite projective dimension $\operatorname{pdim}(M) < \infty$, then $\operatorname{G-dim}(M) = \operatorname{pdim}(M)$ (see [Maş00, Cor. 21]). Consider an I-free approximation (2.4) of an R-module M. In the following we relate the case of minimal Gorenstein dimension of M to Cohen–Macaulayness of V, proving our main result.

Lemma 2.22. *Assume that R is Gorenstein and consider an I-free approximation (2.4) with $W \neq 0$. Then W is a maximal Cohen–Macaulay $\overline{R}$-module if and only if $\operatorname{G-dim}(M) \leq k$. In this case $\operatorname{G-dim}(M) \leq k-1$ if and only if $\operatorname{Ext}_R^k(M,R) = 0$. If $\overline{R}$ is Gorenstein, then $\operatorname{G-dim}(M) \geq k-1$ unless $\overline{\alpha}$ in diagram (2.9) is injective.*

Proof. By hypothesis $M \neq 0$ is finitely generated over the Gorenstein ring R. It follows that (see [Maş00, Thm. 17, Lem. 23.(c)])

$$\operatorname{G-dim}(M) = \max\left\{i \in \mathbb{N} \;\middle|\; \operatorname{Ext}_R^i(M,R) \neq 0\right\} < \infty. \tag{2.19}$$

The Auslander–Bridger Formula (see [Maş00, Thm. 29]) then states that

$$\operatorname{depth}(M) = \operatorname{depth}(R) - \operatorname{G-dim}(M) = \dim(R) - \operatorname{G-dim}(M) = n - \operatorname{G-dim}(M). \tag{2.20}$$

By the Depth Lemma (see [BH93, Prop. 1.2.9]) applied to the short exact sequence (2.1)

$$\begin{aligned}n-k+1 = \operatorname{depth}(\overline{R})+1 &\geq \min\left\{\operatorname{depth}(R), \operatorname{depth}(I)-1\right\}+1 = \operatorname{depth}(I)\\ &\geq \min\left\{\operatorname{depth}(R), \operatorname{depth}(\overline{R})+1\right\} = n-k+1\end{aligned}$$

and hence

$$\operatorname{depth}(IF) = \operatorname{depth}(I) = n-k+1. \tag{2.21}$$

($\Longrightarrow$) Using (2.21) and (2.20) the Depth Lemma applied to the short exact sequence (2.4) gives

$$\operatorname{G-dim}(M) = n - \operatorname{depth}(M) \leq n - \min\left\{\operatorname{depth}(IF), \operatorname{depth}(W)\right\} \leq n-(n-k) = k.$$

($\Longleftarrow$) Using (2.20) and (2.21) the Depth Lemma applied to the short exact sequence (2.4) gives

$$n-k = \dim(\overline{R}) \geq \dim(W) \geq \operatorname{depth}(W) \geq \min\left\{\operatorname{depth}(M), \operatorname{depth}(IF)-1\right\} \geq n-k.$$

By (2.19) this latter inequality becomes $\operatorname{G-dim}(M) \leq k-1$ if and only if $\operatorname{Ext}_R^k(M,R) = 0$ (see [Maş00, Lem. 23.(c)]).

If $\overline{R}$ is Gorenstein and $\overline{\alpha}$ is not injective, then $\operatorname{Ext}_R^{k-1}(M,R) \neq 0$ by Corollary 2.21 and hence $\operatorname{G-dim}(M) \geq k-1$ by (2.19). $\square$

We can now conclude the proof of our main result.

Proof of Theorem 1.2. Since M is I-reflexive, $W = V^{\triangledown}$ by Proposition 2.13.

$(\Longrightarrow)$ Suppose that V is maximal Cohen–Macaulay. Then also W is maximal Cohen–Macaulay and V is (ϖ_R-)reflexive (see [BH93, Prop. 3.3.3.(b).(ii), Thm. 3.3.10.(d).(iii)]). By Corollary 2.21 $\operatorname{Ext}_R^k(M,R)=0$ and by Lemma 2.22 $\operatorname{G-dim}(M)=k-1$.

$(\Longleftarrow)$ Suppose that $\operatorname{G-dim}(M)\le k-1$. By Lemma 2.22 W is maximal Cohen–Macaulay and $\operatorname{Ext}^k(M,R)=0$. By Corollary 2.21 $V=W^\triangledown$ is (ϖ_R-)reflexive and maximal Cohen–Macaulay (see [BH93, Prop. 3.3.3.(b).(ii)]).

The last claim is due to Lemma 2.22. □

2.5. **Restricted I-free approximation.** In this subsection we describe a construction that reduces the support of an I-free approximation (2.4) and preserves I-reflexivity of M under suitable hypotheses. In §3.2 this will be related to the definition of multi-logarithmic differential forms and residues along Cohen–Macaulay spaces (see [Ale14, §10] and [Pol16, Ch. 4]).

Fix an ideal $J\trianglelefteq R$ with $I\subseteq J$ and set $S:=\overline{R}$ and $T:=R/J$. By hypothesis S is Cohen–Macaulay and hence (see [BH93, Prop.1.2.13])

$$\operatorname{Ass}(S)=\operatorname{Min}\operatorname{Spec}(S). \tag{2.22}$$

Lemma 2.23. *There is an inclusion*

$$\operatorname{Supp}_S(T)\cap\operatorname{Ass}(S)\subseteq\operatorname{Ass}_S(T).$$

In particular, equality in $\operatorname{Hom}_S(N,S)$ *for any* T*-module* N*, or in* $\operatorname{Hom}_S(N,T)$ *for any* S*-module* N*, can be checked at* $\operatorname{Ass}_S(T)$.

Proof. The inclusion follows from (2.22) and $\operatorname{Min}\operatorname{Supp}_S(T)\subseteq\operatorname{Ass}_S(T)$. For any T-module N (see [BH93, Exe. 1.2.27])

$$\operatorname{Ass}_S(\operatorname{Hom}_S(N,S))=\operatorname{Supp}_S(N)\cap\operatorname{Ass}(S)\subseteq\operatorname{Supp}_S(T)\cap\operatorname{Ass}(S)\subseteq\operatorname{Ass}_S(T)$$

and the first particular claim follows, the second holds for a similar reason. □

Definition 2.24. For any S-module N we consider the submodule supported on $V(J)$

$$N_T:=\operatorname{Hom}_S(T,N)=\operatorname{Ann}_N(J)\subseteq N.$$

For an I-free approximation (2.4) its *J-restriction* is the I-free approximation

$$0\longrightarrow IF\xrightarrow{\iota_J}M_J\xrightarrow{\rho_T}W_T\longrightarrow 0 \tag{2.23}$$

defined as its image under the map $\operatorname{Ext}_R^1(W,IF)\to\operatorname{Ext}_R^1(W_T,IF)$.

In explicit terms it is the source of a morphism of I-free approximations

$$\begin{tikzcd} 0 \arrow[r] & IF \arrow[r, "\iota"] & M \arrow[r, "\rho"] & W \arrow[r] & 0 \\ 0 \arrow[r] & IF \arrow[r, "\iota_J"] \arrow[u, equal] & M_J \arrow[r, "\rho_T"] \arrow[u, hook] & W_T \arrow[r] \arrow[u, hook] & 0. \end{tikzcd} \tag{2.24}$$

The right square is obtained as the pull-back of ρ and $W_T\hookrightarrow W$, whose universal property applied to ι and $0\colon IF\to W_T$ gives the left square. The analogue of κ in (2.5) for the J-restriction (2.23) is the composition

$$\kappa_J\colon M_J=IF:_M J\subseteq M\xrightarrow{\kappa}F. \tag{2.25}$$

By Lemma 2.2 and the Snake Lemma, applying $-^I$ to (2.24) yields (see Definition 2.8)

$$\begin{tikzcd} 0 & V \arrow[l] \arrow[d, two heads] & F^\vee \arrow[l, "\alpha"'] \arrow[d, equal] & M^I \arrow[l, "\lambda"'] \arrow[d, hook] & 0 \arrow[l] \\ 0 & V^T \arrow[l] & F^\vee \arrow[l, "\alpha^T"'] & M^I_J \arrow[l, "\lambda^J"'] & 0 \arrow[l] \end{tikzcd} \tag{2.26}$$

where the bottom row

$$\begin{tikzcd} 0 & V^T \arrow[l] & F^\vee \arrow[l, "\alpha^T"'] & M^I_J \arrow[l, "\lambda^J"'] & 0 \arrow[l] \end{tikzcd} \tag{2.27}$$

is the I-dual (2.11) of the J-restriction (2.23). In diagram (2.26), we denote

$$U := \ker(V \twoheadrightarrow V^T). \tag{2.28}$$

The J-restriction behaves well under the following hypothesis on T.

$$T_\mathfrak{p} = \begin{cases} S_\mathfrak{p} & \text{if } \mathfrak{p} \in \operatorname{Ass}_S(T), \\ 0 & \text{if } \mathfrak{p} \in \operatorname{Ass}(S) \setminus \operatorname{Ass}_S(T). \end{cases} \tag{2.29}$$

This is due to the following

Remark 2.25. Our constructions commute with localization. As special cases of the J-restriction and its I-dual we record

$$(\iota_J, \rho_T) = \begin{cases} (\iota, \rho) & \text{if } T = S, \\ (\mathrm{id}_{IF}, 0) & \text{if } T = 0, \end{cases} \quad (\lambda^J, \alpha^T) = \begin{cases} (\lambda, \alpha) & \text{if } T = S, \\ (\mathrm{id}_{F^\vee}, 0) & \text{if } T = 0. \end{cases}$$

Localizing (2.24) and (2.26) at the image of $\mathfrak{p} \in \operatorname{Ass}(S)$ under the map $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ yields these special cases under hypothesis (2.29).

In the setup of our applications in §3 condition (2.29) holds true due to the following

Lemma 2.26. *If S is reduced and T is unmixed with $\dim(T) = \dim(S)$, then condition (2.29) holds and $\operatorname{Ass}_S(T) \subseteq \operatorname{Ass}(S)$.*

Proof. By hypothesis on T and (2.22)

$$\operatorname{Ass}_S(T) = \operatorname{Min}\operatorname{Supp}_S(T) \subseteq \operatorname{Min}\operatorname{Spec}(S) = \operatorname{Ass}(S). \tag{2.30}$$

By hypothesis on S, for any $\mathfrak{p} \in \operatorname{Ass}(S)$, $S_\mathfrak{p}$ is a field with factor ring $T_\mathfrak{p}$. If $\mathfrak{p} \in \operatorname{Ass}_S(T)$, then $T_\mathfrak{p} \neq 0$ and hence $T_\mathfrak{p} = S_\mathfrak{p}$. Otherwise, $\mathfrak{p} \notin \operatorname{Supp}_S(T)$ by (2.30) and hence $T_\mathfrak{p} = 0$. □

Lemma 2.27. *Assume that R is Gorenstein and consider the J-restriction (2.23) of an I-free approximation. If T satisfies condition (2.29), then for U as defined in (2.28)*

$$\alpha^{-1}(U) = \{\varphi \in F^\vee \mid \varphi \circ \kappa(M) \subseteq J\omega_R\}.$$

In particular, $JV \subseteq U$.

Proof. Let $\varphi \in F^\vee$ and denote by $\overline{\varphi}_T$ the map $\overline{\varphi}$ in diagram (2.6) for the J-restriction (2.23). Consider the map ψ defined by the commutative diagram

$$\begin{tikzcd} W \arrow[r, "\psi"] \arrow[dr, "\overline{\varphi}"] & T \otimes_R \omega_R \\ W_T \arrow[u, hook] \arrow[r, "\overline{\varphi}_T"] & S \otimes_R \omega_R. \arrow[u, two heads] \end{tikzcd} \tag{2.31}$$

By Lemma 2.23 and since $\omega_R \cong R$ both $\overline{\varphi}_T = 0$ and $\psi = 0$ can be checked at $\operatorname{Ass}_S(T)$. There the vertical maps in diagram (2.31) induce the identity by condition (2.29) and

Remark 2.25. With diagram (2.26), Lemma 2.9 applied to (2.23) and diagram (2.6) it follows that

$$\alpha(\varphi) \in U \iff \alpha^T(\varphi) = 0 \iff \overline{\varphi}_T = 0 \iff \psi = 0 \iff \varphi \circ \kappa(M) \subseteq J\omega_R.$$

This proves the equality and the inclusion follows with $JV = J\alpha(F^\vee) = \alpha(JF^\vee)$. $\square$

Proposition 2.28. *Assume that R is Gorenstein and consider the J-restriction (2.23) of an I-free approximation. If T satisfies condition (2.29), then with M also M_J is I-reflexive.*

Proof. By Lemma 2.27 there is a short exact sequence

$$0 \to U/JV \to V/JV \to V^T \to 0. \tag{2.32}$$

By condition (2.29) and Remark 2.25

$$JS_\mathfrak{p} = \begin{cases} 0 & \text{if } \mathfrak{p} \in \operatorname{Ass}_S(T), \\ S_\mathfrak{p} & \text{if } \mathfrak{p} \in \operatorname{Ass}(S) \setminus \operatorname{Ass}_S(T), \end{cases} \qquad (V \twoheadrightarrow V^T)_\mathfrak{p} = \begin{cases} \mathrm{id}_{V_\mathfrak{p}} & \text{if } \mathfrak{p} \in \operatorname{Ass}_S(T), \\ 0 & \text{if } \mathfrak{p} \in \operatorname{Ass}(S) \setminus \operatorname{Ass}_S(T), \end{cases}$$

and hence

$$\begin{aligned} \forall \mathfrak{p} \in \operatorname{Ass}(S)\colon (JV)_\mathfrak{p} = JS_\mathfrak{p} V_\mathfrak{p} = U_\mathfrak{p} &\implies (U/JV)_\mathfrak{p} = 0 \\ &\implies \dim(U/JV) < \dim(S) = \operatorname{depth}(\overline{\omega}_R). \end{aligned}$$

Then $(U/JV)^\triangledown = 0$ by Ischebeck's Lemma (see [HK71, Satz 1.9]). Using sequence (2.32) and Hom-tensor adjunction it follows that

$$(V^T)^\triangledown = (V/JV)^\triangledown = (T \otimes_S V)^\triangledown = (V^\triangledown)_T.$$

Denote by ν_T the map ν from Lemma 2.9 applied to the J-restriction (2.23). We obtain a diagram

$$\begin{tikzcd} W_T \arrow[rr, "(\nu^\triangledown \circ \delta_{W,\overline{\omega}_R})_T"] \arrow[d, equal] & & (V^\triangledown)_T \arrow[d, equal] \\ W_T \arrow[r, "\delta_{W_T,\overline{\omega}_R}"] & (W_T)^{\triangledown\triangledown} \arrow[r, "(\nu_T)^\triangledown"] & (V^T)^\triangledown. \end{tikzcd} \tag{2.33}$$

By Lemma 2.23 and since $\overline{\omega}_R \cong S$, its commutativity can be checked at $\operatorname{Ass}_S(T)$. By condition (2.29) and Remark 2.25 top and bottom horizontal maps in diagram (2.33) identify at $\operatorname{Ass}_S(T)$. Diagram (2.33) thus commutes and Proposition 2.13 yields the claim. $\square$

The Cohen–Macaulay property is invariant under restriction of scalars $S \to T$ and by Hom-tensor adjunction $\operatorname{Hom}_S(-, \omega_S) = \operatorname{Hom}_T(-, \omega_T)$ on T-modules where (see [BH93, Thm. 3.3.7.(b)])

$$\omega_T = \operatorname{Hom}_S(T, \omega_S). \tag{2.34}$$

Combining Theorem 1.2 and Proposition 2.28 yields the following (see diagram (2.26))

Corollary 2.29. *In addition to the hypotheses of Theorem 1.2, let $J \trianglelefteq R$ with $J \subseteq I$ be such that $T = R/J$ satisfies condition (2.29) and $W_T \neq 0$. Consider the J-restriction (2.23) with I-dual (2.27). Then $W_T = \operatorname{Hom}_T(V^T, \omega_T)$ and V^T is a maximal Cohen–Macaulay T-module if and only if $\operatorname{G-dim}(M_J) \leq k - 1$. In this latter case $V^T = \operatorname{Hom}_T(W_T, \omega_T)$ is ω_T-reflexive. Unless $T \otimes \alpha^T$ (and hence $\overline{\alpha}$) is injective $\operatorname{G-dim}(M_J) \geq k - 1$.* $\square$

Finally we mention a construction analogous to Definition 2.24 not used in the sequel.

Remark 2.30. Assume that J satisfies the hypotheses on I and consider an I-free approximation (2.4) where W is already a T-module. Then $W_T = W$ and $M_J = M$ and the image of (2.4) under the map $\operatorname{Ext}^1_R(W, IF) \to \operatorname{Ext}^1_R(W, JF)$ is a J-free approximation that fits into a commutative diagram with cartesian left square

$$\begin{tikzcd}
0 \arrow[r] & JF \arrow[r] & M^J \arrow[r] & W \arrow[r] & 0 \\
0 \arrow[r] & IF \arrow[r] \arrow[u, hook] & M \arrow[r] \arrow[u, hook] & W \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where $M^J/M_J \cong JF/IF$. In particular, $M^J = M_J$ if and only if $I = J$.

3. Application to logarithmic forms

In this section results from §2 are used to give a more conceptual approach to and to generalize a duality of multi-logarithmic forms found by Pol [Pol16] as a generalization of result by Granger and the first author [GS14].

Let Y be a germ of a smooth complex analytic space of dimension n. Then $Y \cong (\mathbb{C}^n, 0)$ and $\mathscr{O}_Y \cong \mathbb{C}\{x_1, \dots, x_n\}$ by a choice of coordinates $x_1, \dots, x_n$ on Y. We denote by

$$\mathscr{Q}_- := Q(\mathscr{O}_-)$$

the total ring of fractions of $\mathscr{O}_-$. In this section we set $-^* := \operatorname{Hom}_{\mathscr{O}_Y}(-, \mathscr{O}_Y)$.

Let $\Omega^\bullet_Y$ denote the *De Rham algebra on* Y, that is,

$$\mathscr{O}_Y \to \Omega^1_Y, \quad f \mapsto df,$$

is the universally finite $\mathbb{C}$-linear derivation of $\mathscr{O}_Y$ (see [SS72, §2] and [Kun86, §11]) and $\Omega^q_Y = \bigwedge^q_{\mathscr{O}_Y} \Omega^1_Y$ for all $q \geq 0$. In terms of coordinates $\Omega^1_Y \cong \bigoplus_{i=1}^n \mathscr{O}_Y dx_i$ and hence

$$\Omega^q_Y = \bigwedge^q_{\mathscr{O}_Y} \Omega^1_Y \cong \bigoplus_{i_1 < \cdots < i_q} \mathscr{O}_Y dx_{i_1} \wedge \cdots \wedge dx_{i_q}$$

is a free $\mathscr{O}_Y$-module. By definition the dual

$$(\Omega^1_Y)^* = \operatorname{Der}_{\mathbb{C}}(\mathscr{O}_Y) =: \Theta_Y \cong \bigoplus_{i=1}^n \mathscr{O}_Y \frac{\partial}{\partial x_i}$$

is the module of $\mathbb{C}$-*linear derivations on* $\mathscr{O}_Y$, or of vector fields on Y. The module of q-*vector fields on* Y is then the free $\mathscr{O}_Y$-module

$$(\Omega^q_Y)^* = \bigwedge^q_{\mathscr{O}_Y} \Theta_Y =: \Theta^q_Y \cong \bigoplus_{i_1 < \cdots < i_q} \mathscr{O}_Y \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_q}}.$$

Notation 3.1. We set $N := \{1, \dots, n\}$ and $N^q_< := \{\underline{j} \in N^q \mid j_1 < \cdots < j_q\}$. For $\underline{j} \in N^q$ and $\underline{f} = (f_1, \dots, f_\ell) \in \mathscr{O}_Y^\ell$ we abbreviate

$$dx_{\underline{j}} := dx_{j_1} \wedge \cdots \wedge dx_{j_q}, \quad \frac{\partial}{\partial x_{\underline{j}}} := \frac{\partial}{\partial x_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{j_q}},$$
$$\underline{j}_{\hat{i}} := (j_1, \dots, \widehat{j_i}, \dots, j_q), \quad d\underline{f} = df_1 \wedge \cdots \wedge df_\ell.$$

The perfect pairing

$$\Theta^q_Y \times \Omega^q_Y \to \mathscr{O}_Y, \quad (\delta, \omega) \mapsto \langle \delta, \omega \rangle, \tag{3.1}$$

then satisfies

$$\left\langle \frac{\partial}{\partial x_{\underline{j}}}, dx_{\underline{k}} \right\rangle = \delta_{\underline{j},\underline{k}} := \delta_{j_1,k_1} \cdots \delta_{j_q,k_q}. \tag{3.2}$$

3.1. **Log forms along complete intersections.** Let $C \subseteq Y$ be a reduced complete intersection of codimension $k \geq 1$. Then $\mathscr{O}_C = \mathscr{O}_Y/\mathcal{I}_C$ where $\mathcal{I}_C = \mathcal{I}_{C/Y}$ is the ideal of $C \subseteq Y$. Let $\underline{h} = (h_1, \dots, h_k) \in \mathscr{O}_Y^k$ be any regular sequence such that $\mathcal{I}_C = \langle h_1, \dots, h_k \rangle$. Geometrically $C = D_1 \cap \cdots \cap D_k$ where $D_i := \{h_i = 0\}$ for $i = 1, \dots, k$.

Notation 3.2. We denote $D := D_1 \cup \cdots \cup D_k = \{h = 0\}$ where $h := h_1 \cdots h_k$,

$$-(D) := - \otimes_{\mathscr{O}_Y} \mathscr{O}_Y \frac{1}{h}, \qquad -(-D) := - \otimes_{\mathscr{O}_Y} \mathscr{O}_Y h,$$

$$\Sigma = \Sigma_{C/D/Y} := \mathcal{I}_C(D) = \sum_{i=1}^k \frac{h_i}{h} \mathscr{O}_Y \subseteq \mathscr{Q}_Y, \qquad -^\Sigma := \operatorname{Hom}_{\mathscr{O}_Y}(-, \Sigma).$$

Note that $\Sigma = \mathscr{O}_Y$ in case $k = 1$.

The following definition due to Aleksandrov (see [Ale12, §3] and [Pol16, Def. 3.1.4]) generalizes Saito's logarithmic differential forms (see [Sai80]) from the hypersurface to the complete intersection case.

Definition 3.3. The module of *multi-logarithmic differential q-forms on Y along C* is defined by

$$\begin{aligned} \Omega^q(\log C) = \Omega^q_Y(\log C) &:= \left\{ \omega \in \Omega^q_Y \,\middle|\, d\mathcal{I}_C \wedge \omega \subseteq \mathcal{I}_C \Omega^{q+1}_Y \right\}(D) \\ &= \left\{ \omega \in \Omega^q_Y(D) \,\middle|\, \forall i = 1, \dots, k \colon dh_i \wedge \omega \in \Sigma \Omega^{q+1}_Y \right\} \end{aligned}$$

where the equality is due to the Leibniz rule. Observe that

$$\Sigma \Omega^q_Y \subseteq \Omega^q(\log C) \subseteq \mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Omega^q_Y$$

with $\Omega^q(\log C)(-D) \subseteq \mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Omega^q_Y$ independent of D (see [Pol16, Prop. 3.1.10]).

Extending Saito's theory (see [Sai80, §1-2]) Aleksandrov (see [Ale12, §3-4,6]) gives an explicit description of multi-logarithmic differential forms and defines a multi-logarithmic residue map. We summarize his results.

Proposition 3.4. *An element $\omega \in \Omega^q_Y(D)$ lies in $\Omega^q(\log C)$ if and only if there exist $g \in \mathscr{O}_Y$ inducing a non zero-divisor in $\mathscr{O}_C$, $\xi \in \Omega^{q-k}_Y$ and $\eta \in \Sigma\Omega^q_Y$ such that*

$$g\omega = \frac{d\underline{h}}{h} \wedge \xi + \eta.$$

This representation defines a multi-logarithmic residue map

$$\operatorname{res}^q_C \colon \Omega^q(\log C) \to \mathscr{Q}_C \otimes_{\mathscr{O}_C} \Omega^{q-k}_C, \quad \omega \mapsto \frac{\xi}{g},$$

that fits into a short exact multi-logarithmic residue sequence

$$0 \longrightarrow \Sigma\Omega^q_Y \longrightarrow \Omega^q(\log C) \xrightarrow{\operatorname{res}^q_C} \omega^{q-k}_C \longrightarrow 0 \tag{3.3}$$

where ω^p_C is the module of regular meromorphic p-forms on C. ☐

Corollary 3.5. *For $q < k$, $\Omega^q(\log C) = \Sigma\Omega^q_Y$ and $\Omega^n(\log C) = \Omega^n_Y(D)$.* ☐

Remark 3.6. The multi-logarithmic residue map can be written in terms of residue symbols as $\operatorname{res}^q_C(\omega) = \begin{bmatrix} h\omega \\ \underline{h} \end{bmatrix}$ (see [Sch16, §1.2][1]). In particular $\operatorname{res}^k_C(\frac{dh}{h}) = \begin{bmatrix} dh \\ \underline{h} \end{bmatrix} \in \omega^k_C$ is the fundamental form of C (see [Ker83, §5]). $\square$

Higher logarithmic derivation modules play a prominent role in arrangement theory (see for instance [ATW07]). Here we extend the definitions of Granger and the first author (see [GS12, §5]) and by Pol (see [Pol16, Def. 3.2.1]) as follows.

Definition 3.7. We define the module of *multi-logarithmic q-vector fields on Y along C* by

$$\begin{aligned}\operatorname{Der}^q(-\log C) = \operatorname{Der}^q_Y(-\log C) &:= \left\{\delta \in \Theta^q_Y \;\middle|\; \left\langle \delta, \wedge^k d\mathcal{I}_C \wedge \Omega^{q-k}_Y \right\rangle \subseteq \mathcal{I}_C \right\} \\ &= \left\{\delta \in \Theta^q_Y \;\middle|\; \left\langle \delta, d\underline{h} \wedge \Omega^{q-k}_Y \right\rangle \subseteq \mathcal{I}_C \right\}\end{aligned}$$

where the equality is due to the Leibniz rule. Observe that

$$\mathcal{I}_C\Theta^q_Y \subseteq \operatorname{Der}^q(-\log C).$$

Lemma 3.8. *We can identify the functors on $\mathscr{O}_Y$-modules (see Notation 2.1)*

$$-^\Sigma = -(-D)^{\mathcal{I}_C}, \quad (\Sigma \otimes_{\mathscr{O}_Y} -)^\Sigma = -^*,$$

and hence $-^{\Sigma\Sigma} = -^{\mathcal{I}_C\mathcal{I}_C}$.

Proof. Since $\mathscr{O}_Y(D)$ is invertible and by Hom-tensor adjunction

$$-^\Sigma = \operatorname{Hom}_{\mathscr{O}_Y}(-, \mathcal{I}_C(D)) = \operatorname{Hom}_{\mathscr{O}_Y}(-, \operatorname{Hom}_{\mathscr{O}_Y}(\mathscr{O}_Y(-D), \mathcal{I}_C)) = -(-D)^{\mathcal{I}_C}$$

By Lemma 2.3 in case $k \geq 2$, $\mathscr{O}_Y = \mathcal{I}_C^{\mathcal{I}_C} = \Sigma^\Sigma$ and again by Hom-tensor adjunction

$$(\Sigma \otimes_{\mathscr{O}_Y} -)^\Sigma = \operatorname{Hom}_{\mathscr{O}_Y}(\Sigma \otimes_{\mathscr{O}_Y} -, \Sigma) = \operatorname{Hom}_{\mathscr{O}_Y}(-, \Sigma^\Sigma) = -^*. \qquad \square$$

Lemma 3.9. *Any elements $\delta \in \operatorname{Der}^q(-\log C)$ and $\omega \in \Omega^q(\log C)$ pair to $\langle \delta, \omega \rangle \in \Sigma$.*

Proof. Let g, ξ and η be as in Proposition 3.4. Then by definition

$$g\langle \delta, h\omega \rangle = \langle \delta, hg\omega \rangle = \langle \delta, d\underline{h} \wedge \xi + h\eta \rangle = \langle \delta, d\underline{h} \wedge \xi \rangle + h\langle \delta, \eta \rangle \in \mathcal{I}_C.$$

Since g induces a non zero-divisor in $\mathscr{O}_C = \mathscr{O}_Y/\mathcal{I}_C$ this implies that $\langle \delta, h\omega \rangle \in \mathcal{I}_C$ and hence $\langle \delta, \omega \rangle \in \frac{1}{h}\mathcal{I}_C = \Sigma$. $\square$

The following proofs for $q \geq k \geq 1$ proceed along the lines of Saito's base case $q = k = 1$ (see [Sai80, (1.6)]) or Pol's generalization to $q = k \geq 1$ (see [Pol16, Prop. 3.2.13]).

Lemma 3.10. *If $\omega \in \Omega^q_Y(D)$ with $\langle \operatorname{Der}^q(-\log C), \omega \rangle \subseteq \Sigma$, then $\omega \in \Omega^q(\log C)$.*

Proof. For every $\ell \in \{1, \ldots, k\}$ and $\underline{j} \in N^{q+1}_<$ consider

$$\delta^\ell_{\underline{j}} := \sum_{i=1}^{q+1} (-1)^{i+1} \frac{\partial h_\ell}{\partial x_{j_i}} \frac{\partial}{\partial x_{\underline{j}_{\hat{i}}}} \in \Theta^q_Y.$$

For every $\underline{i} \in N^{q-k}$

$$d\underline{h} \wedge dx_{\underline{i}} = \sum_{\underline{k} \in N^q_<} \frac{\partial(\underline{h}, x_{\underline{i}})}{\partial x_{\underline{k}}} dx_{\underline{k}},$$

[1] This remark was made in the first author's talk "Normal crossings in codimension one" at the 2012 Oberwolfach conference "Singularities" (see [Sch12]).

where $\frac{\partial(\underline{h},x_{\underline{i}})}{\partial x_{\underline{k}}}$ is the $q\times q$-minor of the Jacobian matrix of $(\underline{h},x_{\underline{i}})$ with column indices $\underline{k}$, and hence using (3.2)

$$\begin{aligned}\left\langle\delta^\ell_{\underline{j}},d\underline{h}\wedge dx_{\underline{i}}\right\rangle&=\sum_{i=1}^{q+1}(-1)^{i+1}\frac{\partial h_\ell}{\partial x_{j_i}}\sum_{\underline{k}\in N^q_<}\frac{\partial(\underline{h},x_{\underline{i}})}{\partial x_{\underline{k}}}\left\langle\frac{\partial}{\partial x_{\underline{j}_{\hat{i}}}},dx_{\underline{k}}\right\rangle\\&=\sum_{i=1}^{q+1}(-1)^{i+1}\frac{\partial h_\ell}{\partial x_{j_i}}\frac{\partial(\underline{h},x_{\underline{i}})}{\partial x_{\underline{j}_{\hat{i}}}}=\frac{\partial(h_\ell,\underline{h},x_{\underline{i}})}{\partial x_{\underline{j}}}=0.\end{aligned}$$

It follows that $\delta^\ell_{\underline{j}}\in\mathrm{Der}^q(-\log C)$ for all $\ell=1,\dots,k$ and $\underline{j}\in N^{q+1}_<$.

Now let $\omega=\sum_{\underline{k}\in N^q_<}\frac{a_{\underline{k}}}{h}dx_{\underline{k}}\in\Omega^q_Y(D)$ where $a_{\underline{k}}\in\mathscr{O}_Y$. For all $\ell=1,\dots,k$ and $\underline{j}\in N^{q+1}_<$

$$\left\langle\delta^\ell_{\underline{j}},\omega\right\rangle=\sum_{i=1}^{q+1}(-1)^{i+1}\frac{\partial h_\ell}{\partial x_{j_i}}\sum_{\underline{k}\in N^q_<}\frac{a_{\underline{k}}}{h}\left\langle\frac{\partial}{\partial x_{\underline{j}_{\hat{i}}}},dx_{\underline{k}}\right\rangle=\sum_{i=1}^{q+1}(-1)^{i+1}\frac{\partial h_\ell}{\partial x_{j_i}}\frac{a_{\underline{j}_{\hat{i}}}}{h}$$

by (3.2) and hence

$$\begin{aligned}dh_\ell\wedge\omega=\sum_{j=1}^{n}\frac{\partial h_\ell}{\partial x_j}dx_j\wedge\sum_{\underline{k}\in N^q_<}\frac{a_{\underline{k}}}{h}dx_{\underline{k}}&=\sum_{\underline{j}\in N^{q+1}_<}\sum_{i=1}^{q+1}\frac{\partial h_\ell}{\partial x_{j_i}}\frac{a_{\underline{j}_{\hat{i}}}}{h}dx_{j_i}\wedge dx_{\underline{j}_{\hat{i}}}\\&=\sum_{\underline{j}\in N^{q+1}_<}\sum_{i=1}^{q+1}(-1)^{i+1}\frac{\partial h_\ell}{\partial x_{j_i}}\frac{a_{\underline{j}_{\hat{i}}}}{h}dx_{\underline{j}}=\sum_{\underline{j}\in N^{q+1}_<}\left\langle\delta^\ell_{\underline{j}},\omega\right\rangle dx_{\underline{j}}.\end{aligned}$$

If $\langle\mathrm{Der}^q(-\log C),\omega\rangle\subseteq\Sigma$, then $dh_\ell\wedge\omega\in\Sigma\Omega^q_Y$ for all $\ell=1,\dots,k$ and hence $\omega\in\Omega^q(\log C)$. $\square$

Proposition 3.11. *There are chains of $\mathscr{O}_Y$-submodules of $\mathscr{Q}_Y\otimes_{\mathscr{O}_Y}\Omega^q_Y$ and $\mathscr{Q}_Y\otimes_{\mathscr{O}_Y}\Theta^q_Y$*

$$\Omega^q_Y\subseteq\Sigma\Omega^q_Y\subseteq\Omega^q(\log C)\subseteq\Omega^q_Y(D)\subseteq\Sigma\Omega^q_Y(D),\tag{3.4}$$

$$\Sigma\Theta^q_Y\supseteq\Theta^q_Y\supseteq\mathrm{Der}^q(-\log C)\supseteq\mathscr{I}_C\Theta^q_Y\supseteq\Theta^q_Y(-D)\tag{3.5}$$

that are Σ-duals of each other.

Proof. Tensoring with $\mathscr{Q}_Y$ makes both chains collapse. The cokernels of all inclusions are therefore torsion whereas Σ is torsion free. Applying $-^\Sigma$ thus results in a chain of $\mathscr{O}_Y$-modules again. In case of (3.4) this yields

$$(\Omega^q_Y)^\Sigma\supseteq(\Sigma\Omega^q_Y)^\Sigma\supseteq\Omega^q_Y(\log C)^\Sigma\supseteq\Omega^q_Y(D)^\Sigma\supseteq(\Sigma\Omega^q_Y(D))^\Sigma$$

and, with Lemma 3.8 and freeness of Ω^q_Y and Θ^q_Y, the chain of $\mathscr{O}_Y$-submodules of $\mathscr{Q}_Y\otimes_{\mathscr{O}_Y}\Theta^q_Y$

$$\Sigma\Theta^q_Y\supseteq\Theta^q_Y\supseteq\Omega^q_Y(\log C)^\Sigma\supseteq\mathscr{I}_C\Theta^q_Y\supseteq\Theta^q_Y(-D).$$

For every $\delta\in\Omega^q(\log C)^\Sigma$ and $\xi\in\Omega^{q-k}$, $\frac{dh}{h}\wedge\xi\in\Omega^q(\log C)$ by Proposition 3.4, hence

$$\langle\delta,d\underline{h}\wedge\xi\rangle=h\left\langle\delta,\frac{d\underline{h}}{h}\wedge\xi\right\rangle\in h\Sigma=\mathscr{I}_C$$

and $\delta\in\mathrm{Der}^q(-\log C)$. With Lemma 3.9, it follows that $\Omega^q_Y(\log C)^\Sigma=\mathrm{Der}^q(-\log C)$.

By the same reasoning $-^\Sigma$ applied to (3.5) yields a chain of $\mathscr{O}_Y$-modules

$$(\Sigma\Theta^q_Y)^\Sigma\subseteq(\Theta^q_Y)^\Sigma\subseteq\mathrm{Der}^q(-\log C)^\Sigma\subseteq(\Sigma\Theta^q_Y)(-D)^\Sigma\subseteq\Theta^q_Y(-D)^\Sigma$$

that can be rewritten as the chain of $\mathscr{O}_Y$-submodules of $\mathscr{Q}_Y\otimes_{\mathscr{O}_Y}\Omega^q_Y$

$$\Omega^q_Y\subseteq\Sigma\Omega^q_Y\subseteq\mathrm{Der}^q(-\log C)^\Sigma\subseteq\Omega^q_Y(D)\subseteq\Sigma\Omega^q_Y(D).$$

The missing equality $\mathrm{Der}^q(-\log C)^\Sigma = \Omega^q(\log C)$ follows from Lemmas 3.9 and 3.10. $\square$

3.2. **Log forms along Cohen–Macaulay spaces.** Let $X \subseteq Y$ be a reduced Cohen-Macaulay germ of codimension $k \geq 2$. Then $\mathscr{O}_X = \mathscr{O}_Y/\mathcal{I}_X$ where $\mathcal{I}_X := \mathcal{I}_{X/Y}$ denotes the ideal $X \subseteq Y$. There is a reduced complete intersection $C \subseteq Y$ of codimension k such that $X \subseteq C$ and hence $\mathcal{I}_X \supseteq \mathcal{I}_C$ (see [Pol16, Prop. 4.2.1]). Set $X' := \overline{C \setminus X}$ such that $C = X \cup X'$. The link with §2.5 is made by setting

$$S := \mathscr{O}_C, \quad T := \mathscr{O}_X.$$

By Lemma 2.26 condition (2.29) holds and

$$\mathscr{Q}_C = \prod_{\mathfrak{p} \in \mathrm{Ass}_{\mathscr{O}_C}(\mathscr{O}_X)} \mathscr{O}_{X,\mathfrak{p}} \times \prod_{\mathfrak{p} \in \mathrm{Ass}_{\mathscr{O}_C}(\mathscr{O}_{X'})} \mathscr{O}_{X',\mathfrak{p}} = \mathscr{Q}_X \times \mathscr{Q}_{X'}. \tag{3.6}$$

This decomposition extends to differential forms as follows.

Lemma 3.12. *We have $\mathscr{Q}_X d\mathcal{I}_C = \mathscr{Q}_X d\mathcal{I}_X \subseteq \mathscr{Q}_X \otimes_{\mathscr{O}_Y} \Omega^1_Y$ and hence*

$$\mathscr{Q}_C \otimes_{\mathscr{O}_C} \Omega^p_C = \mathscr{Q}_X \otimes_{\mathscr{O}_X} \Omega^p_X \oplus \mathscr{Q}_{X'} \otimes_{\mathscr{O}_{X'}} \Omega^p_{X'}.$$

Proof. By (3.6) we may localize at $\mathfrak{p} \in \mathrm{Ass}_{\mathscr{O}_C}(\mathscr{O}_X)$. We may further assume $p = 1$ since exterior product commutes with extension of scalars. Let $\mathfrak{p} \mapsto \mathfrak{q}$ under $\mathrm{Spec}(\mathscr{O}_C) \to \mathrm{Spec}(\mathscr{O}_Y)$. Then $\mathcal{I}_{C,\mathfrak{q}} = \mathcal{I}_{X,\mathfrak{q}}$ by (3.6) and hence $u\mathcal{I}_X \subseteq \mathcal{I}_C$ for some $u \in \mathscr{O}_Y \setminus \mathfrak{q}$. By the Leibniz rule $ud\mathcal{I}_X \subseteq d\mathcal{I}_C + \mathcal{I}_X du$ and hence the first claim. Since $\Omega^1_C = \Omega^1_Y/(\mathscr{O}_Y d\mathcal{I}_C + \mathcal{I}_C\Omega^1_Y)$ this yields $\Omega^1_{C,\mathfrak{p}} = \Omega^1_{X,\mathfrak{p}}$ and the second claim follows. $\square$

The following fact is well-known (see [Sch16, (2.14)]); we only sketch a proof.

Lemma 3.13. *The modules of regular differential p-forms on X and C are related by $\omega^p_X = \mathrm{Hom}_{\mathscr{O}_C}(\mathscr{O}_X, \omega^p_C) \subseteq \omega^p_C$.*

Proof. Kersken explicitly describes (see [Ker84, (1.2)])

$$\omega^p_X = \left\{ \begin{bmatrix} \xi \\ \underline{h} \end{bmatrix} \;\middle|\; \xi \in \Omega^{p+k}_Y,\ \mathcal{I}_X\xi \subseteq \mathcal{I}_C\Omega^{p+k}_Y,\ d\mathcal{I}_X \wedge \xi \subseteq \mathcal{I}_C\Omega^{p+k+1}_Y \right\} \tag{3.7}$$

where $\begin{bmatrix} \xi \\ \underline{h} \end{bmatrix} = 0$ if and only if $\xi \in \mathcal{I}_C\Omega^{p+k}_Y$. In particular, $\omega^p_X \subseteq \mathrm{Hom}_{\mathscr{O}_C}(\mathscr{O}_X, \omega^p_C) \subseteq \omega^p_C$ and equality in ω^p_C can be checked at $\mathrm{Ass}(\mathscr{O}_C)$. Lemma 3.12 yields the claim. $\square$

The following modules of differential forms on Y due to Aleksandrov (see [Ale14, Def. 10.1] and [Pol16, Def. 4.1.3]) are defined by the relations in (3.7).

Definition 3.14. The module of *multi-logarithmic differential q-forms on Y along X relative to C* is defined by

$$\begin{aligned}\Omega^q(\log X/C) = \Omega^q_Y(\log X/C) &:= \left\{\omega \in \Omega^q_Y \;\middle|\; \mathcal{I}_X\omega \subseteq \mathcal{I}_C\Omega^q_Y,\ d\mathcal{I}_X \wedge \omega \subseteq \mathcal{I}_C\Omega^{q+1}_Y\right\}(D) \\ &= \left\{\omega \in \Omega^q_Y(D) \;\middle|\; \mathcal{I}_X\omega \subseteq \Sigma\Omega^q_Y,\ d\mathcal{I}_X \wedge \omega \subseteq \Sigma\Omega^{q+1}_Y\right\}.\end{aligned}$$

Observe that

$$\Sigma\Omega^q_Y \subseteq \Omega^q(\log X/C) \subseteq \Omega^q(\log C)$$

with $\Omega^q(\log X/C)(-D) \subseteq \mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Omega^q_Y$ independent of D (see [Pol16, Prop. 4.1.5]).

Lemma 3.15. *There is an equality $\Omega^q(\log X/C) = \Sigma\Omega^q_Y :_{\Omega^q(\log C)} \mathcal{I}_X$. In other words, $\Omega^q(\log X/C)(-D) = \mathcal{I}_X\Omega^q_Y :_{\Omega^q(\log C)} \mathcal{I}_X$.*

Proof. There are obvious inclusions

$$\Sigma\Omega^q_Y \subseteq \Omega^q(\log X/C) \subseteq \Sigma\Omega^q_Y :_{\Omega^q(\log C)} \mathcal{I}_X \subseteq \Omega^q(\log C).$$

By Proposition 3.4 and Lemma 3.12

$$\begin{aligned}\omega \in \Sigma\Omega^q_Y :_{\Omega^q(\log C)} \mathcal{I}_X &\implies \mathcal{I}_X \operatorname{res}^q_C(\omega) \subseteq \operatorname{res}^q_C(\Sigma\Omega^q_Y) = 0\\ &\implies \operatorname{res}^q_C(\omega) \in \mathscr{Q}_X \otimes_{\mathscr{O}_X} \Omega^{q-k}_X\\ &\implies 0 = d\mathcal{I}_X \wedge \operatorname{res}^q_C(\omega) = \operatorname{res}^{q+1}_C(d\mathcal{I}_X \wedge \omega)\\ &\implies d\mathcal{I}_X \wedge \omega \subseteq \Sigma\Omega^{q+1}_Y\\ &\implies \omega \in \Omega^q(\log X/C).\end{aligned}$$

□

The idea of Remark 3.6 is used by Aleksandrov (see [Ale14, §10]) to define multi-logarithmic residues along X as the restriction of those along C. The bottom sequence of the diagram in the following Proposition 3.16 appears in his work (see [Ale14, Thm. 10.2]); Pol proved exactness on the right (see [Pol16, Prop. 4.1.21]). An alternative argument is suggested by §2.5. The following data

(3.8)

$$R := \mathscr{O}_Y, \quad I := \mathcal{I}_C, \quad J := \mathcal{I}_X, \quad F := \Omega^q_Y, \quad M := \Omega^q(\log C)(-D), \quad \rho := \frac{1}{h}\operatorname{res}^q_C$$

give rise to an I-free approximation (2.4) with J-restriction (2.23). By Corollary 3.5 $W = 0$ if $q < k$ and (2.4) is trivial for $q = n$. We are therefore concerned with the case $k \leq q < n$. By Lemmas 3.13 and 3.15 (see Definition 2.24 and (2.25))

$$W_T = \omega^{q-k}_X, \quad M_J = \Omega^q(\log X/C)(-D). \tag{3.9}$$

Now twisting diagram (2.24) by D yields the following result.

Proposition 3.16. *Applying* $\operatorname{Ext}^1_{\mathscr{O}_Y}(\omega^{q-k}_X \hookrightarrow \omega^{q-k}_C, \Sigma\Omega^q_Y)$ *to the multi-logarithmic residue sequence* (3.3) *yields a commutative diagram with exact rows and cartesian right square*

$$\begin{tikzcd} 0 \arrow[r] & \Sigma\Omega^q_Y \arrow[r] & \Omega^q(\log C) \arrow[r, "\operatorname{res}^q_C"] & \omega^{q-k}_C \arrow[r] & 0\\ 0 \arrow[r] & \Sigma\Omega^q_Y \arrow[r] \arrow[u, equal] & \Omega^q(\log X/C) \arrow[r, "\operatorname{res}^q_{X/C}"] \arrow[u, hook] & \omega^{q-k}_X \arrow[r] \arrow[u, hook] & 0 \end{tikzcd} \tag{3.10}$$

where ω^p_X *is the module of regular meromorphic* p*-forms on* X. □

3.3. **Higher log vector fields and Jacobian modules.** Pol gives a description of $\operatorname{res}^q_{X/C}$ preserving the analogy with the definition of res^q_C in Proposition 3.4 (see [Pol16, §4.2.1]). As suggested by Remark 3.6 the role of $\frac{dh}{h} \in \Omega^k(\log C)$ is played by a preimage $\frac{\alpha_X}{h} \in \Omega^k(\log X/C)$ of the fundamental form $\begin{bmatrix}\alpha_X\\ \underline{h}\end{bmatrix} \in \omega^0_X$ of X (see [Ker83, §5]).

Definition 3.17. Let $\mathbf{1}_X := (1,0) \in \mathscr{Q}_X \times \mathscr{Q}_{X'} = \mathscr{Q}_C$ (see Lemma 3.12). A *fundamental form of* X *in* Y is an $\alpha_X = \alpha_{X/C/Y} \in \Omega^k_Y$ such that $\overline{\alpha_X} = \mathbf{1}_X d\underline{h} \in \mathscr{Q}_C \otimes_{\mathscr{O}_Y} \Omega^k_Y$.

Such a fundamental form exists and the explicit description of multi-logarithmic differential forms in Proposition 3.4 generalizes verbatim (see [Pol16, Prop. 4.2.6]).

Proposition 3.18. *An element* $\omega \in \Omega^q_Y(D)$ *lies in* $\Omega^q(\log X/C)$ *if and only if there exist* $g \in \mathscr{O}_Y$ *inducing a non zero-divisor in* $\mathscr{O}_C$, $\xi \in \Omega^{q-k}_Y$ *and* $\eta \in \Sigma\Omega^q_Y$ *such that*

$$g\omega = \frac{\alpha_X}{h} \wedge \xi + \eta$$

and the map $\operatorname{res}^q_{X/C}$ *in* (3.10) *is defined by* $\operatorname{res}^q_{X/C}(\omega) = \frac{\xi}{g}$. $\square$

In the same spirit we extend Definition 3.7. We start with the first option as definition.

Definition 3.19. We define the module of *multi-logarithmic q-vector fields on Y along X* by

$$\operatorname{Der}^q(-\log X) = \operatorname{Der}^q_Y(-\log X) := \left\{\delta \in \Theta^q_Y \,\middle|\, \left\langle \delta, \wedge^k d\mathcal{I}_X \wedge \Omega^{q-k}_Y \right\rangle \subseteq \mathcal{I}_X \right\}.$$

The following result completes the analogy with Definition 3.7. In particular $\operatorname{Der}^k(-\log X)$ is Pol's module $\operatorname{Der}^k(-\log X/C)$ (see [Pol16, Def. 4.2.8]) which is thus independent of C.

Lemma 3.20. *We have*

$$\begin{aligned}\operatorname{Der}^q(-\log C) \subseteq \left\{\delta \in \Theta^q_Y \,\middle|\, \left\langle \delta, \alpha_X \wedge \Omega^{q-k}_Y \right\rangle \subseteq \mathcal{I}_X \right\} &= \operatorname{Der}^q(-\log X) \\ = \left\{\delta \in \Theta^q_Y \,\middle|\, \left\langle \delta, \alpha_X \wedge \Omega^{q-k}_Y \right\rangle \subseteq \mathcal{I}_C \right\}&.\end{aligned}$$

Proof. By Definition 3.17 $\overline{\alpha_X} = \overline{\mathbf{1}_X d\underline{h}} = \overline{d\underline{h}} \in \mathscr{Q}_X \otimes_{\mathscr{O}_Y} \Omega^k_Y$. For $\delta \in \Theta^q_Y$ and $\xi \in \Omega^{q-k}_Y$

$$\langle \delta, \alpha_X \wedge \xi \rangle \in \mathcal{I}_X \iff 0 = \overline{\langle \delta, \alpha_X \wedge \xi \rangle} = \left\langle \overline{\delta}, \overline{\alpha_X} \wedge \overline{\xi} \right\rangle = \left\langle \overline{\delta}, \overline{d\underline{h}} \wedge \overline{\xi} \right\rangle = \overline{\langle \delta, d\underline{h} \wedge \xi \rangle} \in \mathscr{Q}_X$$

where $\overline{\delta} \in \mathscr{Q}_X \otimes_{\mathscr{O}_Y} \Theta^q_Y$ and $\overline{\xi} \in \mathscr{Q}_X \otimes_{\mathscr{O}_Y} \Omega^{q-k}_Y$. The claimed inclusion follows. Using the Leibniz rule and that $\mathscr{Q}_X d\mathcal{I}_C = \mathscr{Q}_X d\mathcal{I}_X \subseteq \mathscr{Q}_X \otimes_{\mathscr{O}_Y} \Omega^1_Y$ by Lemma 3.12

$$\begin{aligned}0 = \left\langle \overline{\delta}, \overline{d\underline{h}} \wedge \overline{\xi} \right\rangle \in \mathscr{Q}_X \iff 0 &= \left\langle \overline{\delta}, \wedge^k \overline{d\mathcal{I}_C} \wedge \overline{\xi} \right\rangle = \left\langle \overline{\delta}, \wedge^k \overline{d\mathcal{I}_X} \wedge \overline{\xi} \right\rangle = \overline{\left\langle \delta, \wedge^k d\mathcal{I}_X \wedge \xi \right\rangle} \subseteq \mathscr{Q}_X \\ &\iff \left\langle \delta, \wedge^k d\mathcal{I}_X \wedge \xi \right\rangle \subseteq \mathcal{I}_X.\end{aligned}$$

This proves the first equality. With $\mathcal{I}_C = \mathcal{I}_X \cap \mathcal{I}_{X'}$ the second equality follows from $\alpha_X \in \mathcal{I}_{X'}\Omega^k_Y$ (see [Pol16, Prop. 4.2.5]). $\square$

Using Proposition 3.18 and Lemma 3.20 we obtain the following analogue of Lemma 3.9 and of the equality $\operatorname{Der}^q(-\log C) = \Omega^q(\log C)^\Sigma$ from Proposition 3.11.

Lemma 3.21. *For* $\delta \in \operatorname{Der}^q(-\log X)$ *and* $\omega \in \Omega^q(\log X/C)$ *we have* $\langle \delta, \omega \rangle \in \Sigma$. $\square$

Lemma 3.22. *There is an equality* $\operatorname{Der}^q(-\log X) = \Omega^q(\log X/C)^\Sigma$. $\square$

The following proposition extends Proposition 3.11 and includes the counterpart of Lemma 3.10.

Proposition 3.23. *There are chains of* $\mathscr{O}_Y$*-submodules of* $\mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Omega^q_Y$ *and* $\mathscr{Q}_Y \otimes_{\mathscr{O}_Y} \Theta^q_Y$

$$\Omega^q_Y \subseteq \Sigma\Omega^q_Y \subseteq \Omega^q(\log X/C) \subseteq \Omega^q(\log C) \subseteq \Omega^q_Y(D) \subseteq \Sigma\Omega^q_Y(D),$$
$$\Sigma\Theta^q_Y \supseteq \Theta^q_Y \supseteq \operatorname{Der}^q(-\log X) \supseteq \operatorname{Der}^q(-\log C) \supseteq \mathcal{I}_C\Theta^q_Y \supseteq \Theta^q_Y(-D)$$

that are Σ*-duals of each other.*

Proof. By Lemma 3.8 and Proposition 3.11 M in (3.8) is I-reflexive. By Proposition 2.28 and (3.9) $\Omega^q(\log X/C)(-D)$ is therefore $\mathcal{I}_C$-reflexive and, again by Lemma 3.8, $\Omega^q(\log X/C)$ Σ-reflexive. The claim follows with Proposition 3.11 and Lemmas 3.20 and 3.22. $\square$

Definition 3.24. Contraction with α_X defines a map

$$\alpha^X \colon \Theta^q_Y \to \mathscr{O}_X \otimes_{\mathscr{O}_Y} \Theta^{q-k}_Y = \operatorname{Hom}_{\mathscr{O}_Y}(\Omega^{q-k}_Y, \mathscr{O}_X), \quad \delta \mapsto (\omega \mapsto \overline{\langle \delta, \alpha_X \wedge \omega \rangle}).$$

Taking $p + q = n$ we define the *pth Jacobian module of X* as the $\mathscr{O}_X$-module

$$\mathscr{J}^p_X := \alpha^X(\Theta^q_Y).$$

The Jacobian module $\mathscr{J}_X^{\dim X}$ agrees with Pol's Jacobian ideal $\mathscr{J}_{X/C}$ (see [Pol16, Not. 4.2.14]) which coincides with the ω-Jacobian ideal if X is Gorenstein (see [Pol16, Prop. 4.2.34]).

Remark 3.25. In explicit terms

$$\alpha^X\colon \Theta_Y^q \to \bigoplus_{\underline{i}\in N_<^{q-k}} \mathscr{O}_X dx_{\underline{i}}, \quad \delta \mapsto \sum_{\underline{i}\in N_<^{q-k}} \langle \delta, \alpha_X \wedge dx_{\underline{i}}\rangle dx_{\underline{i}}.$$

In case $X = C$, $\alpha_C = d\underline{h}$ and

$$\langle \delta, d\underline{h}\wedge dx_{\underline{i}}\rangle = \sum_{\underline{j}\in N_<^q} \frac{\partial(\underline{h}, x_{\underline{i}})}{\partial x_{\underline{j}}} \Big\langle \delta, dx_{\underline{j}} \Big\rangle.$$

In particular, $\mathscr{J}_C^{\dim C}$ is the Jacobian ideal of C.

Lemma 3.26. *If $k \le q \le n$, then $\omega_X^{q-k} \neq 0$ and, unless $q = n$, $\mathscr{O}_X \otimes \alpha^X$ is not injective.*

Proof. This can be checked at smooth points of $X = C$ where $\underline{h} = (x_1, \dots, x_k)$ and $\alpha_X = d\underline{h}$. Here $\omega_X^{q-k} = \Omega_X^{q-k} \neq 0$ and $0 \neq \frac{\partial}{\partial x_{\underline{j}}} \in \ker(\mathscr{O}_X \otimes \alpha^X)$ if $\{1, \dots, k\} \not\subseteq \{j_1, \dots, j_q\}$. ☐

By Lemma 3.20 there is a short exact sequence (see [Pol16, Prop. 4.2.16] for $q = k$)

$$0 \longleftarrow \mathscr{J}_X^{n-q} \xleftarrow{\alpha^X} \Theta_Y^q \longleftarrow \mathrm{Der}_Y^q(-\log X) \longleftarrow 0. \tag{3.11}$$

Lemma 3.27. *There is a pairing*

$$\mathscr{J}_X^{n-q} \otimes \omega_X^{q-k} \to \mathrm{Hom}_{\mathscr{O}_C}(\mathscr{O}_X, \mathscr{O}_C)(D) = \omega_X, \quad \Big(\alpha^X(\delta), \mathrm{res}^q_{X/C}(\omega)\Big) \mapsto \langle \delta, \omega\rangle.$$

Proof. By Lemma 3.21 the pairing $\Omega_Y^q(D) \times \Theta_Y^q \to \mathscr{O}_Y(D)$ obtained from (3.1) maps both $\Omega_Y^q(\log X/C) \times \mathrm{Der}_Y^q(-\log X)$ and $\Sigma\Omega_Y^q \otimes \Theta_Y^q$ to Σ. Using the bottom row of (3.10) and (3.11) this yields a pairing $\mathscr{J}_X^{n-q} \otimes \omega_X^{q-k} \to \mathscr{O}_Y(D)/\Sigma = \mathscr{O}_C(D) = \omega_C$. Both $\mathscr{J}_X^{n-q}$ and ω_X^{q-k} are supported on X and applying $\mathrm{Hom}_{\mathscr{O}_C}(\mathscr{O}_X, -)$ yields the claim (see (2.34)). ☐

We can now prove our main application.

Proof of the Theorem 1.3. By Lemmas 3.8 and 3.22 sequence (3.11) in terms of (3.8) is the I-dual J restriction (2.27) twisted by D, that is, $V^T = \mathscr{J}_X^{n-q}$ and $\alpha^T = \alpha^X$ up to a twist by D. With (3.9) and Lemma 3.26 the claim now reduces to Corollary 2.29. The identifications are induced by the pairing in Lemma 3.27. ☐

Proposition 3.28. *The $\mathscr{O}_X$-modules $\mathscr{J}_X^{n-q}$ depend only on X.*

Proof. We identify $\mathscr{J}_X^{n-q} = \Theta_Y^q / \mathrm{Der}_Y^q(-\log X)$ by the exact sequence (3.11). Any isomorphism $Y' \cong Y$ of minimal embeddings of X induces an isomorphism $\varphi\colon \mathscr{O}_Y \cong \mathscr{O}_{Y'}$ over $\mathscr{O}_X$ identifying $\mathcal{I}_{X/Y} \cong \mathcal{I}_{X/Y'}$. There are induced compatible isomorphisms $\Theta_Y^q \cong \Theta_{Y'}^q$ and $\Omega_Y^p \cong \Omega_{Y'}^p$ over φ resulting in an isomorphism over φ

$$\mathrm{Der}_Y^q(-\log X) \cong \mathrm{Der}_{Y'}^q(-\log X).$$

Any general embedding $X \subseteq Y'$ arises from a minimal embedding $X \subseteq Y$ up to isomorphism of the latter as $Y' = Y \times Z$ where $Z \cong (\mathbb{C}^m, 0)$ and hence

$$\mathcal{I}_{X/Y'} = \mathscr{O}_Y \hat{\otimes} \mathfrak{m}_Z + \mathcal{I}_{X/Y} \hat{\otimes} \mathscr{O}_Z.$$

Pick coordinates $z_1, \dots, z_m$ on Z and abbreviate $d\underline{z} := dz_1 \wedge \dots \wedge dz_m$ and $\frac{\partial}{\partial \underline{z}} := \frac{\partial}{\partial z_1} \wedge \dots \wedge \frac{\partial}{\partial z_m}$. Then there are decompositions

$$\Omega_{Y'}^{q+m} = \mathscr{O}_Z \hat{\otimes} \Omega_Y^q \wedge d\underline{z} \oplus \widetilde{\Omega}_{Y'}^{q+m}, \quad \Theta_{Y'}^{q+m} = \mathscr{O}_Z \hat{\otimes} \Theta_Y^q \wedge \frac{\partial}{\partial \underline{z}} \oplus \widetilde{\Theta}_{Y'}^{q+m}$$

where the modules with tilde are generated by basis elements not involving $d\underline{z}$ and $\frac{\partial}{\partial \underline{z}}$ respectively. Fundamental forms of X in Y' and Y can be chosen compatibly as

$$\alpha_{X/C/Y'} = \alpha_{X/C/Y} \wedge d\underline{z} \in \Omega^{k+m}_{Y'}.$$

With Lemma 3.20 this yields inclusions

$$\operatorname{Der}^q_Y(-\log X) \wedge \frac{\partial}{\partial \underline{z}} + \widetilde{\Theta}^{q+m}_{Y'} \subseteq \operatorname{Der}^{q+m}_{Y'}(-\log X) \supseteq \mathcal{I}_{X/Y'}\Theta^{q+m}_{Y'} \supseteq \mathfrak{m}_Z \hat{\otimes} \Theta^q_Y \wedge \frac{\partial}{\partial \underline{z}}$$

and a cartesian square

$$\begin{tikzcd}
\mathscr{O}_Z \hat{\otimes} \Theta^q_Y \arrow[rr, hook, "- \wedge \frac{\partial}{\partial \underline{z}}"] & & \Theta^{q+m}_{Y'} \\
\operatorname{Der}^q_Y(-\log X) + \mathfrak{m}_Z \hat{\otimes} \Theta^q_Y \arrow[u, hook] \arrow[rr, hook] & & \operatorname{Der}^{q+m}_{Y'}(-\log X). \arrow[u, hook]
\end{tikzcd}$$

It gives rise to an isomorphism of $\mathscr{O}_X$-modules

$$\begin{aligned}
\Theta^{q+m}_{Y'} / \operatorname{Der}^{q+m}_{Y'}(-\log X) &\cong \mathscr{O}_Z \hat{\otimes} \Theta^q_Y / (\operatorname{Der}^q_Y(-\log X) + \mathfrak{m}_Z \hat{\otimes} \Theta^q_Y) \\
&\cong \Theta^q_Y / \operatorname{Der}^q_Y(-\log X).
\end{aligned}$$

□

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