QUASIHOMOGENEITY OF CURVES AND THE JACOBIAN ENDOMORPHISM RING

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Abstract. We give a quasihomogeneity criterion for Gorenstein curves. For complete intersections, it is related to the first step of Vasconcelos' normalization algorithm. In the process, we give a simplified proof of the Kunz–Ruppert criterion.

Contents

1. Introduction

We consider a reduced algebroid curve X over an algebraically closed field k of characteristic 0 with coordinate ring A. The Jacobian ideal of X is the 1st Fitting ideal $J_A := F_A^1(\Omega_A^1)$, where Ω_A^1 is the universally finite derivation of A. Lipman [Lip69] showed that X is smooth if and only if J_A is principal. Based on this equivalence, Vasconcelos [Vas91] showed that the normalization of X is obtained by repeatedly replacing A by the endomorphism ring

$$\mathrm{End}_A(J_A^{-1}) = (J_A J_A^{-1})^{-1}.$$

Not much is known about these endomorphism rings. As a coarse measure for the "amount of normality" achieved by this operation we consider the length

$$\rho_X := \ell(\mathrm{End}_A(J_A^{-1})/A).$$

Smoothness of X is equivalent to $\rho_X = 0$ and it is natural to ask when $\rho_X = 1$. We shall answer this question for complete intersection curves.

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By definition, X is *quasihomogeneous* if the kernel of some epimorphism

$$k[[x_1,\dots,x_n]] \twoheadrightarrow A$$

is a quasihomogeneous ideal with respect to some positive weights on the variables.

The main result of this article is the following quasihomogeneity criterion.

Theorem 1.1. *Let X be a non-smooth complete intersection algebroid curve over a field $k=\bar{k}$ with $\operatorname{char} k = 0$. Then X is quasihomogeneous if and only if $\rho_X = 1$.*

The proof of Theorem 1.1 follows from Theorem 3.5 and Corollary 3.7 which rely on a study of semigroups of curves developed in Sections 4 and 5.

2. Fractional ideals

Let $\mathfrak{p}_1,\dots,\mathfrak{p}_r$ be the (minimal) associated primes of A. Then $A_i := A/\mathfrak{p}_i$, $i=1,\dots,r$, are the coordinate rings of the branches of X. By reducedness of A, [HS06, Cor. 2.1.13], Serre's normality criterion and Cohen's structure theorem, the normalization $\tilde{A}$ of A in the total ring of fractions $L := Q(A)$ factorizes as

(2.1)
$$A \hookrightarrow \prod_{i=1}^r A_i \hookrightarrow \prod_{i=1}^r \tilde{A}_i = \tilde{A} \hookrightarrow \prod_{i=1}^r L_i = L, \quad \tilde{A}_i = k[[t_i]], \quad L_i = Q(A_i) = Q(\tilde{A}_i)$$

and we identify A and $\tilde{A}$ with their images in L. For $\alpha = (\alpha_1,\dots,\alpha_r)\in\mathbb{Z}^r$, we shall abbreviate $t^\alpha := (t_1^{\alpha_1},\dots,t_r^{\alpha_r})$, $\partial_t := (\partial_{t_1},\dots,\partial_{t_r})$, and $t^\alpha\partial_t := (t_1^{\alpha_1}\partial_{t_1},\dots,t_r^{\alpha_r}\partial_{t_r})$.

Recall that a *fractional ideal* is a finite A-submodule of L containing a non-zero divisor of A. For any A-module I, we denote the dual by

$$I^{-1} := \operatorname{Hom}_A(I, A)$$

It is easy to prove the following well-known statement.

Lemma 2.1. *For each two fractional ideals I_1 and I_2, also*

$$\operatorname{Hom}_A(I_1,I_2) = \{x\in L \mid xI_1 \subseteq I_2\} \tag{2.2}$$

is again a fractional ideal. In particular, this applies to the dual I^{-1}. Moreover, $I\mapsto I^{-1}$ is inclusion reversing.

Example 2.2.

(1) The maximal ideal $\mathfrak{m}_A$ and the Jacobian ideal J_A of A and the normalization $\tilde{A}$ are fractional ideals.

(2) The conductor

$$C_A := \tilde{A}^{-1}$$

is a fractional ideal by Lemma 2.1 and (1). It is the largest ideal of $\tilde{A}$ which is also an ideal of A. Since $\tilde{A}$ is a product of principal ideal domains and A is not smooth,

$$C_A = \langle t^\delta \rangle \subseteq \mathfrak{m}_A \tag{2.3}$$

is generated by a monomial.

(3) The module

$$M_A := At\partial_t\mathfrak{m}_A$$

is a fractional ideal that is related to quasihomogeneity of X.

(4) For any fractional ideal M, $\operatorname{End}_A(M)$ is a fractional ideal and

$$A \subseteq \operatorname{End}_A(M) \subseteq \tilde{A}$$

since the characteristic polynomial of an endomorphism is a relation of integral dependence by the Cayley–Hamilton theorem.

We shall see that J_A is related to M_A and define

$$\rho'_X := \ell(\operatorname{End}_A(M_A^{-1})/A).$$

Remark 2.3. Note that for X smooth we have $\rho_X = 0 = \rho'_X$.

From now on we assume that X is not smooth.

Lemma 2.4. *For any non-principal fractional ideal I, we have $I^{-1} = \operatorname{Hom}_A(I, \mathfrak{m}_A)$ as fractional ideals. In particular, $\mathfrak{m}_A^{-1} = \operatorname{End}_A(\mathfrak{m}_A)$.*

Proof. After multiplying by a unit in L, we may assume that $I \subseteq A$. Any surjection $\phi\colon I \twoheadrightarrow A$ would have to split as $I = A \cdot x \oplus I'$ with $x \in I$ a non-zero divisor and I' an A-module. For $x' \in I'$, we have $x'x \in Ax \cap I' = 0$ and hence $x' = 0$. But then I would be principal contradicting to the assumption. Thus, any $\phi\colon I \to A$ must map to $\mathfrak{m}_A$ and the first claim follows. The second claim is due our assumption that X is not smooth. □

We denote by $-^\vee := \operatorname{Hom}_A(-, \omega_A^1)$ the dualizing functor. From now on we assume that A is Gorenstein. Then $\omega_A^1 \cong A$ and hence

$$-^{-1} \cong -^\vee \tag{2.4}$$

is an involution on fractional ideals.

Lemma 2.5. *For every fractional ideal I, we have $\operatorname{End}_A(I) = \operatorname{End}_A(I^{-1})$ as fractional ideals.*

Proof. By (2.2), any $\phi \in \operatorname{End}_A(I)$ is just multiplication by some $x \in L$. The same x corresponds to $\phi^{-1} \in \operatorname{End}_A(I^{-1})$ and hence $\operatorname{End}_A(I) \subseteq \operatorname{End}_A(I^{-1})$. By (2.4), the claim follows by applying the above argument to I^{-1} instead of I. □

3. Quasihomogeneity of curves

In the following theorem, we summarize several versions of the Kunz–Ruppert criterion for quasihomogeneity of curves. The original formulation is the equivalence (1) ⇔ (2) ⇔ (3). In the appendix, we comment on a possible issue in its proof by Kunz–Ruppert [KR77] and give a simplified argument.

The equivalence with (4) originates from work of Greuel–Martin–Pfister [GMP85, Satz 2.1] extending the criterion by a numerical characterization of quasihomogeneity, in case of Gorenstein curves. The implication (5) ⇒ (6) in their main result was generalized by Kunz–Waldi [KW88, Thm. 6.21] by comparing two modules:

(1) The module of *Zariski differentials*, which is the reflexive hull $((\Omega_A^1)^{-1})^{-1}$ of Ω_A^1. By the universal property of Ω_A^1, $(\Omega_A^1)^{-1} = \operatorname{Der}_k(A)$ and hence

$$((\Omega_A^1)^{-1})^{-1} = \operatorname{Der}_k(A)^{-1} = (\operatorname{Hom}_A(A\partial_t A, A)\partial_t)^{-1} = ((A\partial_t \mathfrak{m}_A)^{-1}\partial_t)^{-1} = M_A \frac{dt}{t} \tag{3.1}$$

(2) The module of *exact differentials* $c_A dA = \partial_t \mathfrak{m}_A dt$ where

$$c_A \colon \Omega_A^\bullet \to \omega_A^\bullet \tag{3.2}$$

is the *trace map* into the *regular differential forms* on A, which are certain meromorphic forms satisfying $\omega_A^\bullet \cong (\Omega_A^{1-\bullet})^\vee$ (see [Ker84, KW88]).

Theorem 3.1 (Kunz–Ruppert–Waldi)**.** *The following statements are equivalent:*

(1) The curve X is quasihomogeneous.
(2) For some derivation $\chi \in \mathrm{Der}_k(A)$, $A \cdot \chi(A) = \mathfrak{m}_A$.
(3) Multiplication by some unit in $\tilde{A}$ induces an isomorphism $\mathfrak{m}_A \cong t\partial_t\mathfrak{m}_A$.
(4) Every Zariski differential is exact, that is, $t\partial_t\mathfrak{m}_A = M_A$.

Corollary 3.2. *If X is quasihomogeneous then $\rho'_X = 1$.*

Proof. By Theorem 3.1, we have $\mathrm{End}_A(M_A) = \mathrm{End}_A(\mathfrak{m}_A)$ as fractional ideals and hence $\rho'_X = \ell(\mathfrak{m}_A^{-1}/A) = 1$ by Lemma 2.4 and the Gorenstein hypothesis. □

Lemma 3.3. *We have an inclusion* $\mathrm{End}_A(\mathfrak{m}_A) \subseteq \mathrm{End}_A(M_A)$.

Proof. By (3.1) and Lemma 2.5, $\mathrm{Der}_k(A) = M_A^{-1}\partial_t$ and hence $\mathrm{End}_A(M_A) = \mathrm{End}_A(\mathrm{Der}_k(A))$. So, by Lemma 2.4, it suffices to show that $\mathfrak{m}_A^{-1} \subseteq \mathrm{End}_A(\mathrm{Der}_k(A))$. But any $\delta \in \mathrm{Der}_k(A)$ lifts uniquely to $\delta' \in \mathrm{Der}_k(L)$ and such a δ' is in $\mathrm{Der}_k(A)$ exactly if $\delta'(A) \subseteq A$. Now, let $\phi \in \mathfrak{m}_A^{-1}$ be multiplication by $x \in L$. Then $x\delta'(A) \subseteq x\mathfrak{m}_A \subseteq A$ since $\delta'(k) = 0$. The claim follows. □

By Lemma 3.3, we have the following chain of fractional ideals

$$A \subseteq \mathrm{End}_A(\mathfrak{m}_A) \subseteq \mathrm{End}_A(M_A)$$

where the colength of the first inclusion equals $\ell(\mathfrak{m}_A^{-1}/A) = 1$ by Lemma 2.4 and the Gorenstein hypothesis. This yields the following

Proposition 3.4. $\rho'_X = 1$ *implies that* $\mathrm{End}_A(\mathfrak{m}_A) = \mathrm{End}_A(M_A)$.

In Propositions 4.2 and 5.1 in the following sections, we shall see that the equality in Proposition 3.4 implies that in Theorem 3.1.(4). Combined with Corollary 3.2 this proves the following statement.

Theorem 3.5. *A Gorenstein curve X is quasihomogeneous if and only if $\rho'_X = 1$.*

Recall the following result from [Sch13] which uses the coincidence of the Jacobian and the ω-Jacobian ideal for complete intersections (see [OZ87, §3]).

Proposition 3.6. *If X is a complete intersection then $\omega_A^0 \cong J_A^{-1}$.*

Proof. By hypothesis and [Pie79, Prop. 1], (3.2) induces a surjection $\Omega_A^1 \twoheadrightarrow J_A$ with torsion kernel. The claim follows by dualizing. □

In the situation of Proposition 3.6, (3.1) yields

$$J_A \cong (\omega_A^0)^{-1} \cong ((\Omega_A^1)^\vee)^{-1} \cong ((\Omega_A^1)^{-1})^{-1} = M_A$$

and, by Lemma 2.5, we deduce the following statement.

Corollary 3.7. *If X is a complete intersection curve then* $\mathrm{End}_A(J_A) = \mathrm{End}_A(M_A)$ *and, in particular, $\rho_X = \rho'_X$.*

4. Semigroups

Let $\nu_i\colon L_i \to \mathbb{Z} \cup \{\infty\}$ be the discrete valuation with respect to the parameter t_i and define the *multivaluation* on L to be

$$\nu := (\nu_1, \dots, \nu_r) : L \to (\mathbb{Z} \cup \{\infty\})^r.$$

Let $D(L) := \{x \in L \mid \forall i = 1, \dots, r : x_i \neq 0\}$ denote the set of non-zero divisors in L and set $D(M) := D(L) \cap M$ for any subset $M \subseteq L$. Note that $D(A) := A \setminus \bigcup_{i=1}^r \mathfrak{p}_i$.

Definition 4.1. For any subset $M \subseteq L$, we set $\Gamma(M) := \nu(M \cap D(L))$. Then the *semigroup of A* is defined as

$$\Gamma_A := \Gamma(A) \subset \mathbb{N}^r.$$

Note that, for any fractional ideal I, $\Gamma(I)$ is a Γ_A*-set*, that is,

$$\alpha \in \Gamma_A, \beta \in \Gamma(I) \Rightarrow \alpha + \beta \in \Gamma(I).$$

Although, in general, $\mathfrak{m}_A$ and $t\partial_t\mathfrak{m}_A$ are incomparable and $\mathfrak{m}_A \not\cong M_A$ (see appendix), we have at least $t\partial_t\mathfrak{m}_A \subseteq M_A$ and

$$\Gamma(\mathfrak{m}_A) = \Gamma(t\partial_t\mathfrak{m}_A) \subseteq \Gamma(M_A). \tag{4.1}$$

The purpose of this section is to prove the following result.

Proposition 4.2. *If $\Gamma(\mathfrak{m}_A) = \Gamma(M_A)$ then $t\partial_t\mathfrak{m}_A = M_A$.*

By (2.3), we have that

$$C_A \subseteq t\partial_t\mathfrak{m}_A$$

Thus, the proof of Proposition 4.2 follows from equation (4.1) and the following lemma.

Lemma 4.3. *Let $M \subseteq N \subseteq L$ be two k-vector subspaces of $\tilde{A}$ and $\delta \in \mathbb{N}_+^r$ such that $t^\delta \tilde{A} \subseteq M$. Then the equality $\Gamma(M) = \Gamma(N)$ implies $M = N$.*

Proof. Let $x = (x_1, \dots, x_r)$ be an element of N. If $\nu_i(x) \geq \delta_i$ for all i, then already $x \in t^\delta \tilde{A} \subseteq M$.

We can eliminate the zero components of x by choosing $y \in t^\delta \tilde{A} \subseteq M$ such that $x + y \in D(N)$ and hence $\nu_i(x + y) \geq \min(\nu_i(x), \delta_i)$ for all i. If for some i we have $\nu_i(x) < \delta_i$, there is by hypothesis an element $x' \in D(M)$ such that $\nu(x') = \nu(x+y)$ and $\nu_i(x+y-x') > \nu_i(x+y)$ as well as $\nu_j(x+y-x') \geq \nu_j(x+y)$ for all $j \neq i$. We can then conclude by an induction on $\sum_{i=0}^r \max\{\delta_i - \nu_i(x), 0\}$ that $x+y-x' \in M$ and hence $x \in M$. □

Remark 4.4. By Lemma 4.3, $\delta' + \mathbb{N}^r \subseteq \Gamma_A$ implies that $\delta \leq \delta'$ for δ as in (2.3).

5. Gorenstein symmetry

The purpose of this section is to prove the following result.

Proposition 5.1. *If $\operatorname{End}_A(\mathfrak{m}_A) = \operatorname{End}_A(M_A)$ then $\Gamma(\mathfrak{m}_A) = \Gamma(M_A)$.*

The proof of Proposition 5.1 uses that X being Gorenstein is equivalent to a symmetry property of Γ_A which is due to Kunz [Kun70] in the irreducible case and to Delgado [DdlM88] in general. The formulation of the precise statement requires some notation. Recall that $\Gamma(C_A)=\delta+\mathbb{N}^r$ by (2.3) and we set $\tau=\delta-(1,\dots,1)$. For any $\alpha\in\mathbb{Z}^r$, we denote

$$\Delta(\alpha):=\bigcup_{i=1}^r\Delta_i(\alpha),\quad \Delta_i(\alpha):=\{\beta\in\mathbb{Z}^r\mid\alpha_i=\beta_i\text{ and }\alpha_j<\beta_j\text{ if }j\neq i\}.$$

Definition 5.2. The semigroup Γ_A is called *symmetric* if

$$\forall\alpha\in\mathbb{Z}^r:\alpha\in\Gamma_A\Leftrightarrow\Delta(\tau-\alpha)\cap\Gamma_A=\emptyset. \tag{5.1}$$

Theorem 5.3. *(Delgado) The curve X is Gorenstein if and only if its semigroup Γ_A is symmetric.*

Remark 5.4. In the irreducible case $r=1$ the symmetry condition of Delgado reduces to the classical Kunz symmetry condition

$$\forall\alpha\in\{0,\dots,\tau\}:\alpha\in\Gamma_A\Leftrightarrow\tau-\alpha\notin\Gamma_A.$$

We prove Proposition 5.1 in a sequence of lemmas.

Lemma 5.5.
(1) $\Gamma_A\subseteq\{0\}\cup((1,\dots,1)+\mathbb{N}^r)$.
(2) $\tau\in\Gamma_A$ *if and only if* $r>1$.
(3) $\Delta(\tau)\cap\Gamma_A=\emptyset$.

Proof.
(1) Let $\alpha\in\Gamma_A$ be such that $\alpha_i=0$ for some i. Then $\alpha=\nu(x)$ for some $x\in A$ with $x_i\notin\mathfrak{m}_{A_i}$. This implies that $x\notin\mathfrak{m}_A$ and hence $\alpha=0$.
(2) By (1), $\Delta(0)\cap\Gamma_A=\emptyset$ if and only if $r>1$ and the claim follows from (5.1).
(3) This follows from $0\in\Gamma_A$ and (5.1).
□

Remark 5.6. For any $\beta\in\mathbb{N}^r$, $\tau+\beta\notin\Gamma_A$ if and only if β has exactly one zero component, generalizing Lemma 5.5.(3).

Lemma 5.7.
(1) *If* $\Gamma(\mathfrak{m}_A)\subsetneq\Gamma(M_A)$ *then* $\Delta(\tau)\cap\Gamma(M_A)\neq\emptyset$.
(2) *If* $\Delta(\tau)\cap\Gamma(M_A)\neq\emptyset$ *then* $\Delta_i(\tau)\subseteq\Gamma(M_A)$ *for some* i.

Proof.
(1) Let $\alpha\in\Gamma(M_A)\setminus\Gamma(\mathfrak{m}_A)$. Then, by (5.1) and (4.1), there is a $\beta\in\Delta_i(\tau-\alpha)\cap\Gamma_A\subseteq\Gamma(\mathfrak{m}_A)\subseteq\Gamma(M_A)$. Thus, $\alpha+\beta\in\Delta_i(\tau)\cap\Gamma(M_A)\subseteq\Delta(\tau)\cap\Gamma(M_A)$.
(2) We may assume that there is an element $x\in M_A$ with $\nu(x)\in\Delta_1(\tau)\cap\Gamma(M_A)$. Up to a factor in k^*, $x\equiv(t^{\tau_1},\dots)\mod t^\delta\tilde{A}$. For any $\beta\in\Delta_1(\tau)$, $x-t^\beta\in t^\delta\tilde{A}\subseteq M_A$ and hence $t^\beta\in M_A$ and $\beta\in\Gamma(M_A)$.
□

Lemma 5.8.
(1) $\Gamma(\mathrm{End}_A(\mathfrak{m}_A))\setminus\Gamma_A=\Delta(\tau)$.
(2) *If* $\Delta_i(\tau)\subseteq\Gamma(M_A)$ *for some* i *then* $(\Gamma(\mathrm{End}_A(M_A))\setminus\Gamma_A)\cap\bigcup_{j<0}je_i+\Delta_i(\tau)\neq\emptyset$.

Proof.

(1) By Lemma 5.5.(1), $\Delta(\tau)+\Gamma(\mathfrak{m}_A)\subseteq\delta+\mathbb{Z}^r\subseteq\Gamma(\mathfrak{m}_A)$ and hence $\supseteq$ by Lemma 4.3 and 5.5.(3). To prove $\subseteq$, let $\alpha\in\Gamma(\mathrm{End}_A(\mathfrak{m}_A))\setminus\Gamma_A$. Then, by (5.1), there is a $\beta\in\Delta(\tau-\alpha)\cap\Gamma_A$ and hence $\alpha+\beta\in\Delta(\tau)$. As $\beta\in\Gamma(\mathfrak{m}_A)$ leads to the contradiction $\alpha+\beta\in\Delta(\tau)\cap\Gamma(\mathfrak{m}_A)=\emptyset$ by Lemma 5.5.(3), we must have $\beta=0$ and hence $\alpha\in\Delta(\tau)$.

(2) There exists a minimal $m\le 0$ such that $\bigcup_{m<j\le 1} je_i+\Delta_i(\tau)\subseteq\Gamma(\mathfrak{m}_A)$. In fact, $m\ge-\tau_i$ by Lemma 5.5.(1). By (4.1) and the hypothesis, $\delta+me_i+\mathbb{N}^r\subseteq\Gamma(M_A)$ and hence $\alpha+\Gamma(M_A)\subseteq\Gamma(\mathfrak{m}_A)\subseteq\Gamma(M_A)$ for any $\alpha\in me_i+\Delta_i(\tau)\setminus\Gamma_A$ by Lemma 5.5.(1). This implies $t^\alpha\in\mathrm{End}_A(M_A)$ by Lemma 4.3. □

Proof of Proposition 5.1. Assuming that $\Gamma(\mathfrak{m}_A)\subsetneq\Gamma(M_A)$, Lemma 5.7 applies followed by Lemma 5.8. The conclusions of the latter show that

$$\Gamma(\mathrm{End}_A(\mathfrak{m}_A))\subsetneq\Gamma(\mathrm{End}_A(M_A))$$

and hence that $\mathrm{End}_A(\mathfrak{m}_A)\subsetneq\mathrm{End}_A(M_A)$ by Lemma 4.3. □

Appendix: Kunz-Ruppert criterion

In the process of proving the implication (2) ⇒ (1) in Theorem 3.1, Kunz and Ruppert seem to claim (see [KR77, page 6, line 2]) and use that

$$A\cdot t\partial_t(A)\cong\mathfrak{m}_A \tag{5.2}$$

as A-submodule of $\tilde{A}$. The following is a counter-example for this statement.

By abuse of notation, we denote $\mathfrak{m}_{\widetilde{A}}:=\mathfrak{m}_{\widetilde{A}_1}\times\cdots\times\mathfrak{m}_{\widetilde{A}_r}$.

Example 5.9. Consider the (non-quasihomogeneous) plane curve singularity defined by $x^4+xy^4+y^5=0$. After a coordinate change, the equation reads

$$f:=x^4-y(x+y)^4=0.$$

Then the normalization $A=k[[x,y]]/\langle f\rangle\subset\widetilde{A}=k[[t]]$ is given by

$$x=\frac{t^5}{1-t}=t^5+t^6+t^7+\cdots,\quad y=t^4.$$

On the other hand, the left hand side of (5.2) considered modulo $\mathfrak{m}_{\widetilde{A}}^8\supset\mathfrak{m}_A\cdot t\partial_t(A)$ is the k-vector space generated by the 7-jets

$$\widetilde{x}=t\partial_t(x)=\eta\cdot x\equiv 5t^5+6t^6+7t^7\mod\mathfrak{m}_{\widetilde{A}}^8,\quad\widetilde{y}=t\partial_t(y)=4y, \tag{5.3}$$

where

$$\eta:=\frac{5-4t}{1-t}.$$

If there were an isomorphism (5.2) then, both sides being fractional ideals, it would have to be induced by multiplication by some unit $\varepsilon\in\widetilde{A}^*$ with

$$\varepsilon\equiv\eta\equiv 5+t+t^2\mod\mathfrak{m}_{\widetilde{A}}^3.$$

Note that the 3-jet

$$\varepsilon\equiv 5+t+t^2+\alpha t^3\mod\mathfrak{m}_{\widetilde{A}}^4,\quad\alpha\in k,$$

determines the 7-jet

$$\varepsilon\cdot y\equiv 5t^4+t^5+t^6+\alpha t^7\mod\mathfrak{m}_{\widetilde{A}}^8.$$

But for no choice of α this expression lies in the k-span of (5.3). Therefore, there is no isomorphism (5.2) for the curve under consideration.

The following Proposition 5.11 contains the statement of [KR77, Satz 2.2], which yields the implication (2) $\Rightarrow$ (1) in Theorem 3.1.

Remark 5.10. Let $\chi \in \mathrm{Der}_k(A)$. By Scheja–Wiebe [SW73, (2.5)], $\chi(\mathfrak{p}_i) \subset \mathfrak{p}_i$ and hence χ induces a derivation $\chi_i \in \mathrm{Der}_k(A_i)$. As A_i is a domain, Seidenberg [Sei66] shows that χ_i lifts to a derivation $\widetilde{\chi}_i \in \mathrm{Der}_k(\widetilde{A}_i)$. So by (2.1), $\widetilde{\chi} := (\widetilde{\chi}_1, \dots, \widetilde{\chi}_r) \in \mathrm{Der}_k(\widetilde{A})$ is a lift of χ. As χ extend uniquely to any localization and hence to L, $\widetilde{\chi}$ is unique. While this proves part 1) of [KR77, Satz 2.2], it is actually not needed.

Recall [SW73, page 168, Def.], that a derivation $\delta \in \mathrm{Der}_k(A)$ is called diagonalizable if $\mathfrak{m}_A$ is generated by eigenvectors of δ.

Proposition 5.11. *Any $\chi \in \mathrm{Der}_k(A)$ satisfying (2) in Theorem 3.1 lifts uniquely to $\widetilde{\chi} \in \mathrm{Der}_k(\widetilde{A})$ such that*

$$\widetilde{\chi} = \gamma \cdot t\partial_t \tag{5.4}$$

for some

$$\gamma \in k^r \tag{5.5}$$

after a suitable coordinate change. Moreover, χ is diagonalizable with non-zero eigenvalues on $\mathfrak{m}_A$ and can be chosen with eigenvalues in $\mathbb{N}_+$.

Proof. The k-derivation χ lifts uniquely to a k-derivation $\widetilde{\chi} = (\widetilde{\chi}_1, \dots, \widetilde{\chi}_r) \in \mathrm{Der}_k(L)$. By finiteness of the normalization, $0 \neq x_i \in \mathfrak{m}_{\widetilde{A}_i}$ for some $x = (x_1, \dots, x_r) \in \mathfrak{m}_A$. Choosing $x \in \mathfrak{m}_A$ with $\nu_i(x_i)$ minimal yields $\nu_i(\widetilde{\chi}_i(x_i)) \geq \nu_i(x_i)$. By (2) in Theorem 3.1, equality holds for some such choice of $x \in \mathfrak{m}_A$. Hence $\widetilde{\chi}_i = \gamma_i \cdot t_i\partial_{t_i}$ for some $\gamma_i \in \widetilde{A}_i^*$ and (5.4) is obtained by setting $\gamma := (\gamma_1, \dots, \gamma_r) \in \widetilde{A}^*$.

In order to achieve (5.5) by a coordinate change, apply the Poincaré–Dulac decomposition theorem (see [AA88, Ch. 3. §3.2] or [Sai71, Satz 3]) to $\delta = \widetilde{\chi}_i$. Then

$$\delta = \sigma + \eta, \quad \sigma = \gamma(0) \cdot t\partial_t, \quad \eta \in \mathrm{End}_k(\mathfrak{m}_{\widetilde{A}_i}/\mathfrak{m}^2_{\widetilde{A}_i}) \text{ nilpotent}, \quad [\sigma, \eta] = 0,$$

and hence $\eta = 0$. Finally, the conductor is a $\widetilde{\chi}$-invariant (see (2.3)) finite-codimensional k-subspace of $\widetilde{A}$ contained in $\mathfrak{m}_A$ which yields the last statement (see [KR77, page 7] for details). ∎

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