

A normal form algorithm for the Brieskorn lattice

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Abstract

This article describes a normal form algorithm for the Brieskorn lattice of an isolated hypersurface singularity. It is the basis of efficient algorithms to compute the Bernstein–Sato polynomial, the complex monodromy, and Hodge–theoretic invariants of the singularity such as the spectral pairs and good bases of the Brieskorn lattice. The algorithm is a variant of Buchberger’s normal form algorithm for power series rings using the idea of partial standard bases and adic convergence replacing termination.

Key words: hypersurface singularity, Brieskorn lattice, Bernstein-Sato polynomial, monodromy, spectral pairs, good basis, mixed Hodge structure, standard basis
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1 Introduction

Isolated hypersurface singularities form the simplest class of singularities. Their intensive study in the past has led to a variety of invariants. The Milnor number is one of the simplest, and can easily be computed using standard basis methods. A finer invariant is the monodromy of the singularity. E. Brieskorn [1] developed the theoretical background for computing the complex monodromy. He gave an ad hoc definition of an object H'' , later called the Brieskorn lattice. Its great importance was a priori not clear. The complex monodromy can be expressed in terms of the differential structure of the Brieskorn lattice.

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The finest known invariants come from a mixed Hodge structure associated to an isolated hypersurface singularity. The notion of a mixed Hodge structure was introduced by P. Deligne [2] as a generalization of the classical Hodge structure on the cohomology of a compact Kähler manifold. J.H.M. Steenbrink [3] defined this mixed Hodge structure in terms of resolutions of singularities. A.N. Varchenko [4] and later J. Scherk and J.H.M. Steenbrink [5] described this mixed Hodge structure in terms of the differential structure of the Brieskorn lattice. The mixed Hodge numbers correspond to the spectral pairs and determine the complex monodromy. The spectral pairs have a semicontinuity property [6] with respect to unfoldings of the singularity.

Based on properties of the mixed Hodge structure, M. Saito [7] constructed two endomorphisms A_0 and A_1 of the Milnor algebra. These two endomorphisms determine the differential structure of the Brieskorn lattice and, immediately, the above invariants.

The Bernstein–Sato polynomial is associated to a general complex polynomial [8] or convergent power series [9]. T. Oaku [10] presented the first algorithm to compute it in the global case. A new method by M. Noro [11] is impressively faster. In the isolated singularity case, B. Malgrange [12] described the Bernstein–Sato polynomial in terms of the Brieskorn lattice and its close relation to the complex monodromy.

In [13–16], we have developed algorithmic methods to compute all of the above invariants of isolated hypersurface singularities. There is an implementation [17,18] of these algorithms in the computer algebra system SINGULAR [19]. Our algorithm to compute the complex monodromy is much faster and can compute much more difficult examples than Brieskorn’s algorithm. Our algorithm to compute the local Bernstein–Sato polynomial is based on B. Malgrange’s description in terms of the Brieskorn lattice. It is much faster than M. Noro’s algorithm since computations in rings of differential operators are replaced by computing their action on power series rings. However, it is restricted to the isolated singularity case while M. Noro’s algorithm works in the general global case. All our algorithms require the computation of a basis representation in the Brieskorn lattice. In [13, Sec. 10.2] this is done by a sequence of divisions by the Jacobian ideal which is, in general, very hard to compute. The subject of this article is a normal form algorithm for the Brieskorn lattice replacing this sequence of full divisions by a sequence of partial divisions. This new method turns out to be much more efficient.

In the first section, we recall the definition and the main properties of the Brieskorn lattice. We introduce the formal Brieskorn lattice and describe it as a cokernel of a formal family of differential operators which is finite over the base.

In the second section, we consider such a formal family of differential operators in general. We describe a normal form algorithm to compute a presentation of the cokernel which is a finitely generated module over the formal power series ring in the parameters of the family. This algorithm is a variant of B. Buchberger's [20,21] normal form algorithm. There are three major differences compared to the classical algorithm:

- (1) The polynomial ring is replaced by a formal power series ring. Termination of the algorithm is replaced by adic convergence.
- (2) The standard basis is replaced by a partial standard basis, a set of power series which specializes to a standard basis.
- (3) There is only a module structure with respect to the parameters of the family and the partial standard basis is not finite.

Although the algorithm does not terminate in general, it serves to compute exact results by using appropriate degree bounds. The algorithms in [13,16] implicitly contain such degree bounds to compute the above invariants of isolated hypersurface singularities. There are also a priori degree bounds in [15], but they are useless in practice. Essentially the double number of variables plus the double Milnor number is a degree bound that satisfies all requirements.

In the third section, we demonstrate the power of our algorithm. We apply the SINGULAR implementation [18] to examples from [13,16] and [11] and list the timings.

Families which are finite over the base occur in many situations in algebraic geometry and singularity theory. For example, A. Frühbis-Krüger [22] has developed algorithms to compute moduli spaces and adjacencies of singularities based on the idea of partial standard bases. One can expect more applications of our methods in the future.

We shall denote row vectors by a lower bar, column vectors by an upper bar, row indices by lower indices, and column indices by upper indices.

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2 The formal Brieskorn lattice

Let $f : U \rightarrow \mathbb{C}$ be a holomorphic function on an open neighbourhood $0 \in U \subset \mathbb{C}^n$ of the origin. We choose a system of complex coordinates $\underline{x} = x_1, \dots, x_n$ at $0 \in \mathbb{C}^n$ and denote by $\underline{\partial} = \partial_1, \dots, \partial_n = \partial_{x_1}, \dots, \partial_{x_n}$ the corresponding derivatives such that the commutator of ∂_i and x_j is $[\partial_i, x_j] = \delta_{i,j}$. We consider f as a germ of a holomorphic function at $0 \in \mathbb{C}^n$, which means that U can be arbitrarily small. This is equivalent to considering the convergent power series $f \in \mathbb{C}\{\underline{x}\}$. We assume that $f(0) = 0$, and that the origin is an isolated critical point of f . This means that $0 \in U$ is the only point with $\partial_1(f)(0) = \dots = \partial_n(f)(0) = 0$ for some U , or, more algebraically, that $\langle \underline{x} \rangle^m \subset \langle \underline{\partial}(f) \rangle \subset \langle \underline{x} \rangle$ for some $m \geq 1$. The complex dimension

$$\mu = \dim_{\mathbb{C}}(\mathbb{C}\{\underline{x}\} / \langle \underline{\partial}(f) \rangle) < \infty$$

is called the Milnor number. By the finite determinacy theorem [23, Thm. 9.1.4], one can choose, in this case, a coordinate system \underline{x} such that $f \in \mathbb{C}[\underline{x}]$ is a polynomial.

We denote by $\Omega^\bullet = \Omega_{\mathbb{C}^n, 0}^\bullet$ the complex of germs of holomorphic differential forms at $0 \in \mathbb{C}^n$. Its elements are differential forms with coefficients in the convergent power series ring $\mathbb{C}\{\underline{x}\}$. The Brieskorn lattice [1] is defined by

$$H'' = \Omega^n / df \wedge d\Omega^{n-2}$$

and becomes a $\mathbb{C}\{t\}$ -module by setting

$$t \cdot [\omega] = [f\omega] \tag{1}$$

for $[\omega] \in H''$. By M. Sebastiani [24], H'' is a free $\mathbb{C}\{t\}$ -module of rank μ . We denote by Ω the μ -dimensional \mathbb{C} -vector space

$$\Omega = \Omega^n / df \wedge \Omega^{n-1} \cong \mathbb{C}\{\underline{x}\} / \langle \underline{\partial}(f) \rangle.$$

The operators d and $df = df \wedge \cdot$ define two exact sequences.

Lemma 1 (Poincaré Lemma)

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}\{\underline{x}\} \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega^n \longrightarrow 0$$

is an exact sequence of \mathbb{C} -vector spaces.

Since completion is exact, Lemma 1 remains valid when replacing Ω^\bullet by its $\langle \underline{x} \rangle$ -adic completion $\widehat{\Omega}^\bullet$. The elements of $\widehat{\Omega}^\bullet$ are differential forms with coefficients in the formal power series ring $\mathbb{C}[[\underline{x}]]$.

Lemma 2 (De Rham Lemma)

$$0 \longrightarrow \mathbb{C}\{\underline{x}\} \xrightarrow{df} \Omega^1 \xrightarrow{df} \dots \xrightarrow{df} \Omega^n \longrightarrow \Omega \longrightarrow 0$$

is an exact sequence of $\mathbb{C}\{\underline{x}\}$ -modules.

Also Lemma 2 remains valid when replacing Ω^\bullet by $\widehat{\Omega}^\bullet$. From Lemma 1 and 2 follows that one can define a \mathbb{C} -linear operator s on H'' by

$$s \cdot [d\eta] = [df \wedge \eta] \quad (2)$$

for $[d\eta] \in H''$. From Lemma 1 follows that s is injective. The image of s is $sH'' = df \wedge \Omega^n / df \wedge d\Omega^{n-2}$ and hence

$$H''/sH'' = \Omega.$$

Also s defines a module structure on H'' over a power series ring. This power series ring is the ring

$$\mathbb{C}\{\{s\}\} = \left\{ \sum_{i=0}^{\infty} a_i s^i \in \mathbb{C}[[s]] \mid \sum_{i=0}^{\infty} \frac{a_i}{i!} t^i \in \mathbb{C}\{t\} \right\} \subset \mathbb{C}[[s]]$$

of microlocal operators with constant coefficients and, by F. Pham [25], H'' is a free $\mathbb{C}\{\{s\}\}$ -module of rank μ . From the definitions of t and s follows immediately that the commutator of t and s is

$$[t, s] = s^2.$$

We define a \mathbb{C} -linear operator ∂_s on the localization $H'' \otimes_{\mathbb{C}\{\{s\}\}} \mathbb{C}\{\{s\}\}[s^{-1}]$ by

$$t = s^2 \partial_s. \quad (3)$$

Then t is a differential operator on H'' with respect to the $\mathbb{C}\{\{s\}\}$ -structure. There is also \mathbb{C} -linear operator ∂_t on the localization $H'' \otimes_{\mathbb{C}\{t\}} \mathbb{C}\{t\}[t^{-1}]$ defined by

$$s = \partial_t^{-1}. \quad (4)$$

Then the commutator of ∂_t and t is $[\partial_t, t] = 1$ and hence ∂_t is a derivative by t .

Definition 3

- (1) We call the topology induced by the $\langle \underline{x} \rangle$ -adic topology on Ω^n on the quotient H'' the $\langle \underline{x} \rangle$ -adic topology on H'' .
- (2) We call the completion \widehat{H}'' of H'' with respect to the $\langle \underline{x} \rangle$ -adic topology the formal Brieskorn lattice.

Lemma 4 *There is a natural isomorphism*

$$\widehat{H}'' = \widehat{\Omega}^n / df \wedge d\widehat{\Omega}^{n-2}.$$

Proof. By definition,

$$\widehat{H}'' = \varprojlim_k \left(\Omega^n / \left(\langle \underline{x} \rangle^k \Omega^n + df \wedge d\Omega^{n-2} \right) \right).$$

Since $\underline{0} \in \mathbb{C}^n$ is a critical point of f , $\langle \partial(f) \rangle \subset \langle \underline{x} \rangle$ and hence

$$\begin{aligned} df \wedge d(\langle \underline{x} \rangle^k \Omega^{n-2}) &\subset \langle \underline{x} \rangle^k \Omega^n, \\ df \wedge d(\langle \underline{x} \rangle^k \widehat{\Omega}^{n-2}) &\subset \langle \underline{x} \rangle^k \widehat{\Omega}^n. \end{aligned}$$

Since $\Omega^n / \langle \underline{x} \rangle^k \Omega^n = \widehat{\Omega}^n / \langle \underline{x} \rangle^k \widehat{\Omega}^n$, this implies that

$$\Omega^n / \left(\langle \underline{x} \rangle^k \Omega^n + df \wedge d\Omega^{n-2} \right) = \widehat{\Omega}^n / \left(\langle \underline{x} \rangle^k \widehat{\Omega}^n + df \wedge d\widehat{\Omega}^{n-2} \right)$$

defines a natural isomorphism of inverse systems. Hence,

$$\widehat{H}'' = \varprojlim_k \left(\widehat{\Omega}^n / \left(\langle \underline{x} \rangle^k \widehat{\Omega}^n + df \wedge d\widehat{\Omega}^{n-2} \right) \right) = \widehat{\Omega}^n / df \wedge d\widehat{\Omega}^{n-2}. \quad \square$$

The following theorem [1, Prop. 3.3] is essential for Brieskorn's algorithm to compute the complex monodromy, which is based on the $\mathbb{C}\{t\}$ -structure of the Brieskorn lattice.

Theorem 5 *The $\langle t \rangle$ -adic and $\langle \underline{x} \rangle$ -adic topology on H'' coincide. In particular, the $\langle t \rangle$ -adic completion of H'' is naturally isomorphic to \widehat{H}'' and \widehat{H}'' is a free $\mathbb{C}[[t]]$ -module of rank μ .*

The $\mathbb{C}\{\{s\}\}$ -structure of the Brieskorn lattice is more algebraic and, therefore, more appropriate for computational purposes. The following proposition [16, Prop. 7] is the analogue of theorem 5 for the $\mathbb{C}\{\{s\}\}$ -structure, but it is much easier to prove.

Proposition 6 *The $\langle s \rangle$ -adic and $\langle \underline{x} \rangle$ -adic topology on H'' coincide. In particular, the $\langle s \rangle$ -adic completion of H'' is naturally isomorphic to \widehat{H}'' and \widehat{H}'' is a free $\mathbb{C}[[s]]$ -module of rank μ .*

Proof. We denote

$$\begin{aligned} d\underline{x} &= dx_1 \wedge \cdots \wedge dx_n, \\ d\underline{x}_i &= dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n \end{aligned}$$

for $1 \leq i \leq n$. Let

$$\left[g \partial_i(f) d\underline{x} \right] \in \left(\langle \partial(f) \rangle^{2k} d\underline{x} + df \wedge d\Omega^{n-2} \right) / df \wedge d\Omega^{n-2} \subset H''$$

for some $k \geq 1$. By (2),

$$\begin{aligned} \left[g \partial_i(f) d\underline{x} \right] &= \left[(-1)^{i+1} df \wedge (g d\underline{x}_i) \right] \\ &= s \left[(-1)^{i+1} d(g d\underline{x}_i) \right] \\ &= s \left[\partial_i(g) d\underline{x} \right] \\ &\in s \left(\langle \partial(f) \rangle^{2(k-1)} d\underline{x} + df \wedge d\Omega^{n-2} \right) / df \wedge d\Omega^{n-2} \end{aligned}$$

and hence, by induction,

$$\left(\langle \partial(f) \rangle^{2k} d\underline{x} + df \wedge d\Omega^{n-2} \right) / df \wedge d\Omega^{n-2} \subset s^k H''.$$

Since $\underline{0} \in \mathbb{C}^n$ is an isolated critical point of f , $\langle \underline{x} \rangle^m \subset \langle \partial(f) \rangle \subset \langle \underline{x} \rangle$ for some $m \geq 1$ and hence

$$\begin{aligned} \left(\langle \underline{x} \rangle^{2km} d\underline{x} + df \wedge d\Omega^{n-2} \right) / df \wedge d\Omega^{n-2} \\ \subset \left(\langle \partial(f) \rangle^{2k} d\underline{x} + df \wedge d\Omega^{n-2} \right) / df \wedge d\Omega^{n-2}. \end{aligned}$$

This implies that

$$\begin{aligned} \left(\langle \underline{x} \rangle^{2km} d\underline{x} + df \wedge d\Omega^{n-2} \right) / df \wedge d\Omega^{n-2} \subset s^k H'' \\ \subset \left(\langle \underline{x} \rangle^k d\underline{x} + df \wedge d\Omega^{n-2} \right) / df \wedge d\Omega^{n-2}. \end{aligned}$$

Hence, the $\langle s \rangle$ -adic and $\langle \underline{x} \rangle$ -adic topology on H'' coincide. \square

Note that the formal Brieskorn lattice is a (t, s) -module in the sense of D. Barlet [26,27]. The following proposition [16, Prop. 8] describes the $\mathbb{C}[[s]]$ -module \widehat{H}'' as a quotient of the power series ring $\mathbb{C}[[s, \underline{x}]]$. It will lead to a normal form algorithm for \widehat{H}'' in the next section.

Proposition 7 $d\underline{x}$ induces a $\mathbb{C}[[s]]$ -isomorphism

$$\widehat{H}'' = \widehat{\Omega}^n[[s]] / (df - s d) \widehat{\Omega}^{n-1}[[s]] \xleftarrow{\sim} \mathbb{C}[[s, \underline{x}]] / \langle \partial(f) - s \partial \rangle \mathbb{C}[[s, \underline{x}]].$$

Proof. Since

$$df \wedge d\widehat{\Omega}^{n-2} = (df - s d) d\widehat{\Omega}^{n-2} \subset (df - s d) \widehat{\Omega}^{n-1}[[s]]$$

and by Lemma 4 and (2), there is a natural $\mathbb{C}[[s]]$ -linear map

$$\widehat{H}'' \xrightarrow{\phi} \widehat{\Omega}^n[[s]] / (df - sd)\widehat{\Omega}^{n-1}[[s]].$$

Let $\omega = \sum_{k \geq 0} \omega_k s^k \in \widehat{\Omega}^{n-1}[[s]]$ with $(df - sd)\omega \in \widehat{\Omega}^n$. Then $df \wedge \omega_{k+1} = d\omega_k$ and hence, by (2),

$$s[d\omega_{k+1}] = [df \wedge \omega_{k+1}] = [d\omega_k] \in \widehat{H}''$$

for all $k \geq 0$. In particular, $[d\omega_0] \in \bigcap_{k \geq 0} s^k \widehat{H}'' = \{0\}$ and hence, by Lemma 4,

$$d\omega_0 \in df \wedge d\widehat{\Omega}^{n-2} = d(df \wedge \widehat{\Omega}^{n-2}).$$

By Lemma 1, this implies that $\omega_0 \in d\widehat{\Omega}^{n-2} + df \wedge \widehat{\Omega}^{n-2}$ and hence

$$(df - sd)\omega = df \wedge \omega_0 \in df \wedge d\widehat{\Omega}^{n-2}.$$

This shows that

$$(df - sd)\widehat{\Omega}^{n-1}[[s]] \cap \widehat{\Omega}^n = df \wedge d\widehat{\Omega}^{n-2}$$

and hence, by Lemma 4, that ϕ is injective. By Lemma 1, $d\widehat{\Omega}^{n-1} = \widehat{\Omega}^n$, and hence ϕ is surjective.

For $\eta = \sum_{i=1}^n (-1)^{i+1} g_i d\widehat{x}_i \in \widehat{\Omega}^{n-1}[[s]]$,

$$(df - sd)\eta = \sum_{i=1}^n (\partial_i(f)g_i - s\partial_i(g_i)) d\mathbf{x} = (\partial(f) - s\partial)\bar{g} d\mathbf{x}.$$

Hence, $d\mathbf{x}$ induces a $\mathbb{C}[[s]]$ -isomorphism

$$\widehat{\Omega}^n[[s]] / (df - sd)\widehat{\Omega}^{n-1}[[s]] \xleftarrow[\sim]{d\mathbf{x}} \mathbb{C}[[s, \mathbf{x}]] / \langle \partial(f) - s\partial \rangle \mathbb{C}[[s, \mathbf{x}]]. \quad \square$$

Proposition 7 is the starting point for more general considerations in the next section.

3 Formal families of differential operators

Let \mathbb{K} be a computable field and $\underline{F} = F_1, \dots, F_r \in \mathbb{K}[[\underline{s}, \underline{x}]] \langle \underline{\partial} \rangle$ a formal family of differential operators where $\underline{x} = x_1, \dots, x_n$, $\underline{\partial} = \partial_1, \dots, \partial_n = \partial_{x_1}, \dots, \partial_{x_n}$, and $\underline{s} = s_1, \dots, s_m$. Note that the elements of $\mathbb{K}[[\underline{s}, \underline{x}]] \langle \underline{\partial} \rangle$ are polynomial in $\underline{\partial}$.

The brackets $\langle \cdot \rangle$ indicate that the commutator $[x_i, \partial_i] = \delta_{i,j}$ is not zero. We want to compute the cokernel $H = \mathbb{K}[\underline{s}, \underline{x}] / \langle \underline{F} \rangle \mathbb{K}[\underline{s}, \underline{x}]$ of the $\mathbb{K}[\underline{s}]$ -linear map

$$\mathbb{K}[\underline{s}, \underline{x}]^r \xrightarrow{\underline{F}} \mathbb{K}[\underline{s}, \underline{x}] \xrightarrow{\pi_H} H \longrightarrow 0.$$

We assume that the specialization

$$\underline{f} = f_1, \dots, f_r = \underline{F}(\underline{s} = 0) \in \mathbb{K}[\underline{x}]$$

is independent of $\underline{\partial}$ and that $\langle \underline{x} \rangle^k \mathbb{K}[\underline{x}] \subset \langle \underline{f} \rangle \mathbb{K}[\underline{x}]$ for some $k \geq 0$. In particular,

$$\mu = \dim_{\mathbb{K}}(\mathbb{K}[\underline{x}] / \langle \underline{f} \rangle \mathbb{K}[\underline{x}]) = \dim_{\mathbb{K}}(H / \langle \underline{s} \rangle H) < \infty$$

and hence H is a finitely generated $\mathbb{K}[\underline{s}]$ -module. Then there is a matrix $D = (\underline{d}^j)_j \in \mathbb{K}[\underline{s}, \underline{x}] \langle \underline{\partial} \rangle^{m \times r}$ such that

$$\underline{F} = \underline{f} - \underline{s}D.$$

Our considerations are motivated by the following special case.

Remark 8 *By Proposition 7, for $\mathbb{K} = \mathbb{C}$, $m = 1$, $r = n$, $\underline{f} = \underline{\partial}(f)$, and $D = \underline{\partial}$,*

$$H \cong_{\mathbb{C}[\underline{s}]} \widehat{H}''$$

is the formal Brieskorn lattice.

Let $<_{\underline{x}}$ be a local degree ordering with respect to a weighted degree $\deg_{\underline{x}}$ on the set of monomials $\{x^{\underline{\alpha}} \mid \underline{\alpha} \in \mathbb{N}^n\} = \mathbb{N}^n$ of $\mathbb{K}[\underline{x}]$. This means that

$$\mathbb{N}^n \xrightarrow{\deg_{\underline{x}}} \mathbb{Q}$$

is a semigroup homomorphism with $\deg_{\underline{x}}(x_i) < 0$, and that $<_{\underline{x}}$ is a semigroup ordering such that

$$\deg_{\underline{x}}(x^{\underline{\alpha}}) < \deg_{\underline{x}}(x^{\underline{\beta}}) \Rightarrow x^{\underline{\alpha}} <_{\underline{x}} x^{\underline{\beta}}.$$

The support of $p = \sum_{\underline{\alpha}} p_{\underline{\alpha}} x^{\underline{\alpha}} \in \mathbb{K}[\underline{x}]$ is defined by $\text{supp}(p) = \{\underline{\alpha} \in \mathbb{N}^n \mid p_{\underline{\alpha}} \neq 0\}$. We denote the leading exponent, resp. leading term, with respect to $<_{\underline{x}}$ by lexp , resp. lead . This means that

$$\begin{aligned} \text{lexp}(p) &= \max_{<_{\underline{x}}} \text{supp}(p), \\ \text{lead}(p) &= p_{\text{lexp}(p)} x^{\text{lexp}(p)} \end{aligned}$$

for $p = \sum_{\underline{\alpha}} p_{\underline{\alpha}} x^{\underline{\alpha}} \in \mathbb{K}[\underline{x}]$ and that

$$\text{lead}(P) = \{\text{lead}(p) \mid p \in P\}$$

for a subset $P \subset \mathbb{K}[\underline{x}]$. Note that the maximum exists by Dickson's Lemma [28, Lem. 1.2.6]. The weighted degree $\deg_{\underline{x}}$ extends to $\mathbb{K}[\underline{x}]$ by setting

$$\deg_{\underline{x}}(p) = \deg \text{lexp}(p)$$

for $p \in \mathbb{K}[\underline{x}]$. Since $[\partial_i, x_i] = 1$, $\deg_{\underline{x}}$ extends to $\mathbb{K}[\underline{x}]\langle \partial \rangle$ by setting

$$\deg_{\underline{x}}(\partial_i) = -\deg_{\underline{x}}(x_i) > 0.$$

Let $g = g_1, \dots, g_l$ be a standard basis of $\langle f \rangle \mathbb{K}[\underline{x}]$. This means that $0 \neq g_i \in \langle f \rangle \mathbb{K}[\underline{x}]$ and

$$\text{lead}(\langle f \rangle \mathbb{K}[\underline{x}]) = \langle \text{lead}(g) \rangle \mathbb{K}[\underline{x}] \quad (5)$$

which implies that

$$\langle f \rangle \mathbb{K}[\underline{x}] = \langle g \rangle \mathbb{K}[\underline{x}] \quad (6)$$

by the division theorem. Let

$$\underline{m} = (m_i)_{i=1, \dots, \mu} = (\underline{x}^\beta)_{\underline{x}^\beta \notin \langle \text{lead}(g) \rangle \mathbb{K}[\underline{x}]}$$

be increasingly ordered with respect to $<_{\underline{x}}$. Then

$$\mathbb{K}[\underline{x}] = \langle \underline{m} \rangle \mathbb{K} \oplus \langle \text{lead}(g) \rangle \mathbb{K}[\underline{x}] \quad (7)$$

and hence, by (5) and (6), $\mathbb{K}[\underline{x}] = \langle \underline{m} \rangle \mathbb{K} \oplus \langle g \rangle \mathbb{K}[\underline{x}]$. Then \underline{m} represents a \mathbb{K} -basis of

$$H / \langle s \rangle H = \mathbb{K}[\underline{x}] / \langle g \rangle \mathbb{K}[\underline{x}] = \langle \underline{m} \rangle \mathbb{K} \quad (8)$$

and, by Nakayama's Lemma, \underline{m} represents a minimal set of $\mathbb{K}[\underline{s}]$ -generators of H . Note that if H is free then it is free of rank μ . Let $U = (\bar{u}_i)_i \in \mathbb{K}[\underline{x}]^{r \times l}$ be a matrix such that

$$\underline{g} = \underline{f}U.$$

Remark 9 *If $\underline{f} \in \mathbb{K}[\underline{x}]$ then one can compute \underline{g} and U with coefficients in $\mathbb{K}[\underline{x}]$ by Lazard's method based on Buchberger's standard basis algorithm [29, Lem. 1.7] and homogenization. In general, the power series $f_i \in \mathbb{K}[[\underline{x}]]$ can be represented by generating functions $\mathbb{N}^n \rightarrow \mathbb{K}$ and Buchberger's standard basis algorithm with respect to a local degree ordering computes such generating functions for \underline{g} and U .*

Let $<_{\underline{s}}$ be a local degree ordering with respect to a weighted degree $\deg_{\underline{s}}$ on the monomials of $\mathbb{K}[[\underline{s}]]$. Let

$$< = (<_{\underline{s}}, <_{\underline{x}})$$

be the block ordering of $<_{\underline{s}}$ and $<_{\underline{x}}$ on the monomials of $\mathbb{K}[[\underline{s}, \underline{x}]]$ and

$$\deg = \deg_{\underline{s}} + \deg_{\underline{x}}$$

the sum of the weighted degrees $\deg_{\underline{s}}$ and $\deg_{\underline{x}}$. This means that

$$\underline{s}^{\alpha'} \underline{x}^{\beta'} < \underline{s}^{\alpha''} \underline{x}^{\beta''} \Leftrightarrow \underline{s}^{\alpha'} <_{\underline{s}} \underline{s}^{\alpha''} \vee (\underline{s}^{\alpha'} = \underline{s}^{\alpha''} \wedge \underline{x}^{\beta'} <_{\underline{x}} \underline{x}^{\beta''})$$

and

$$\deg(\underline{s}^{\alpha} \underline{x}^{\beta}) = \deg_{\underline{s}}(\underline{s}^{\alpha}) + \deg_{\underline{x}}(\underline{x}^{\beta}).$$

As before, we denote the leading exponent, resp. leading term, with respect to $<$ by lexp , resp. lead , and extend \deg to $\mathbb{K}[[\underline{s}, \underline{x}]]\langle \partial \rangle$. Note that $<$ is not a degree ordering with respect to \deg . This means that

$$\deg \text{lead} \neq \deg.$$

We denote by the leading exponent, resp. leading term, with respect to the partial ordering $<_{\underline{s}}$ on $\mathbb{K}[[\underline{s}, \underline{x}]]$ by $\text{lexp}_{\underline{s}}$, resp. $\text{lead}_{\underline{s}}$, and the partial degree $<_{\underline{s}}$ on $\mathbb{K}[[\underline{s}, \underline{x}]]$ by $\deg_{\underline{s}}$. This means that

$$\begin{aligned} \text{lexp}_{\underline{s}}(p) &= \max_{<_{\underline{s}}} \pi_{\mathbb{N}^m}(\text{supp}(p)), \\ \text{lead}_{\underline{s}}(p) &= p_{\text{lexp}_{\underline{s}}(p)} \underline{s}^{\text{lexp}_{\underline{s}}(p)}, \\ \deg_{\underline{s}}(p) &= \deg \text{lexp}_{\underline{s}}(p) \end{aligned}$$

for $p = \sum_{\alpha} p_{\alpha} \underline{s}^{\alpha} \in \mathbb{K}[[\underline{s}, \underline{x}]]$ where $\pi_{\mathbb{N}^m} : \mathbb{N}^m \times \mathbb{N}^n \longrightarrow \mathbb{N}^m$ is the canonical projection. We denote by $\min \deg$, resp. $\max \deg$, the minimal, resp. the maximal, degree of the components of a vector or a matrix.

Let

$$\underline{G} = G_1, \dots, G_l = \underline{F}U = \underline{g} - \underline{s}DU.$$

In the special fibre $\underline{s} = \underline{0}$, \underline{G} induces the standard basis \underline{g} . We call \underline{G} a partial standard basis of the formal family \underline{F} .

The following example is taken from [16, Sec. 8].

Example 10 Let $\mathbb{K} = \mathbb{C}$, $\underline{F} = \partial(f) - s\partial$ as in Remark 8, and $f = x^5 + x^2y^2 + y^5$. Note that f defines a $T_{2,5,5}$ singularity at the origin. Let $<_{(x,y)}$ be the local degree ordering with $\deg(x) = \deg(y) = -1$ and $x > y$. Then one computes

$$\begin{aligned} \underline{f} &= 2xy^2 + 5x^4, 2x^2y + 5y^4, \\ \underline{g} &= 2x^2y + 5y^4, 2xy^2 + 5x^4, 5x^5 - 5y^5, 10y^6 + 25x^3y^4, \\ \mu &= 11, \\ \underline{m} &= y^5, y^4, y^3, y^2, xy, y, x^4, x^3, x^2, x, 1, \\ U &= \begin{pmatrix} 0 & 1 & x & -2xy \\ 1 & 0 & -y & 2y^2 + 5x^3 \end{pmatrix} \end{aligned}$$

and hence

$$\begin{aligned} \underline{G} = & 2x^2y + 5y^4 - s\partial_y, 2xy^2 + 5x^4 - s\partial_x, \\ & 5x^5 - 5y^5 - sx\partial_x + sy\partial_y, 10y^6 + 25x^3y^4 + 2sxy\partial_x - s(2y^2 + 5x^3)\partial_y. \end{aligned}$$

We denote by $\underline{F}\underline{x}^{\mathbb{N}^n} = (F_i\underline{x}^\alpha)_{i,\alpha}$ the generators of the $\mathbb{C}[[s]]$ -module $\langle \underline{F} \rangle \mathbb{K}[[s, \underline{x}]]$.

Lemma 11 *H is a free $\mathbb{K}[[s]]$ -module if and only if $\underline{G}\underline{x}^{\mathbb{N}^n}$ is a standard basis of the $\mathbb{K}[[s]]$ -module $\langle \underline{F} \rangle \mathbb{K}[[s, \underline{x}]]$.*

Proof. By (8) and Nakayama's Lemma, \underline{m} represents a minimal set of generators of H . Since $H = \mathbb{K}[[s, \underline{x}]] / \langle \underline{F} \rangle \mathbb{K}[[s, \underline{x}]]$,

$$\mathbb{K}[[s, \underline{x}]] = \langle \underline{m} \rangle \mathbb{K}[[s]] + \langle \underline{F} \rangle \mathbb{K}[[s, \underline{x}]]$$

and H is free if and only if

$$\langle \underline{F} \rangle \mathbb{K}[[s, \underline{x}]] \cap \langle \underline{m} \rangle \mathbb{K}[[s]] = 0.$$

By (5) and (7), this is equivalent to

$$\text{lead}(\langle \underline{F} \rangle \mathbb{K}[[s, \underline{x}]] \cap \langle \underline{m} \rangle \mathbb{K}[[s]]) = \langle \text{lead}(\underline{g}) \rangle \mathbb{K}[[s]] = \langle \text{lead}(\underline{G}\underline{x}^{\mathbb{N}^n}) \rangle \mathbb{K}[[s]]. \quad \square$$

By Proposition 7,

$$\mathbb{C}[[s, \underline{x}]] / \langle \partial(f) - s\partial \rangle \mathbb{C}[[s, \underline{x}]] \cong_{\mathbb{C}[[s]]} H''$$

is a free $\mathbb{C}[[s]]$ -module of rank μ . We shall now give an elementary proof of this fact.

Proposition 12 *If $\mathbb{K} = \mathbb{C}$ and $\underline{F} = \partial(f) - s\partial$ as in Remark 8 then H is a free $\mathbb{C}[[s]]$ -module of rank μ .*

Proof. Let $0 \neq p \in \langle \underline{m} \rangle \mathbb{C}[[s]] \cap \langle \partial(f) - s\partial \rangle \mathbb{C}[[s, \underline{x}]]$. Then $\text{lead}(p) \in \langle \underline{m} \rangle \mathbb{C}[[s]]$ and $p = (\partial(f) + s\partial)\bar{q}$ for some $\bar{q} \in \mathbb{C}[[s, \underline{x}]]$ with maximal $\max \deg_s(\bar{q})$. By (5) and (7), this implies that $\partial(f)\text{lead}_s(\bar{q}) = 0$ and hence, by Lemma 2, we may assume that there are $1 \leq i < j \leq n$, $k \geq 0$, and $r \in \mathbb{C}[[\underline{x}]]$ such that

$$\text{lead}_s(\bar{q}) = s^k r (\partial_i(f)\bar{e}_j - \partial_j(f)\bar{e}_i).$$

This implies that

$$\partial \text{lead}_s(\bar{q}) = s^k \partial(f) (\partial_j(r)\bar{e}_i - \partial_i(r)\bar{e}_j)$$

and hence

$$p = (\underline{\partial}(f) - s\underline{\partial})(\bar{q} - \text{lead}_s(\bar{q}) - s^{k+1}(\partial_j(r)\bar{e}_i - \partial_i(r)\bar{e}_j)).$$

This is a contradiction to the maximality of $\max \deg_s(\bar{q})$. Hence,

$$\langle \underline{m} \rangle \mathbb{C}[[s]] \cap \langle \underline{\partial}(f) - s\underline{\partial} \rangle \mathbb{C}[[s, \underline{x}]] = 0$$

and H is free. \square

Our aim is now to define a filtration $V = (V_K)_{K \leq 0}$ on $\mathbb{K}[[s, \underline{x}]]$ by $\mathbb{K}[[s]]$ -modules which is

- (1) a basis of the $\langle \underline{s}, \underline{x} \rangle$ -adic topology on $\mathbb{K}[[s, \underline{x}]]$,
- (2) compatible with reduction with respect to the partial standard basis \underline{G} ,
- (3) mapped by π_H onto the basis $(\langle \underline{s} \rangle^K H)_{K \geq 0}$ of the $\langle \underline{s} \rangle$ -adic topology on H .

This will lead to a normal form algorithm for \underline{H} .

For a given weighted degree $\deg_{\underline{x}}$, let the weighted degree $\deg_{\underline{s}}$ be such that

$$\deg(s_j) \leq \min \deg(\underline{m}) + \min \deg(\underline{x}) - \max \deg(\underline{d}^j). \quad (9)$$

Let the strictly increasing sequence $N = (N_K)_{K \leq 0}$ be defined by

$$N_K = -K \min \deg(\underline{s}) - \min \deg(\underline{x}) + \max \deg(D). \quad (10)$$

Let $V = (V_K)_{K \leq 0}$ be the strictly increasing filtration on $\mathbb{K}[[s, \underline{x}]]$ by $\mathbb{K}[[s]]$ -modules

$$V_K = \{p \in \mathbb{K}[[s, \underline{x}]] \mid \deg(p) < N_K\} + \langle \underline{s} \rangle^{-K} \mathbb{K}[[s, \underline{x}]]. \quad (11)$$

Remark 13 For $\underline{F} = \underline{\partial}(f) - s\underline{\partial}$ as in Remark 8, we can choose

$$\begin{aligned} \deg(s) &= \min \deg(\underline{m}) + 2 \min \deg(\underline{x}), \\ N_K &= -K \deg(s) - 2 \min \deg(\underline{x}). \end{aligned}$$

Example 14 In example 10, $\deg(s) = -7$ and $N_K = 7K + 2$.

The following proposition is a generalization of [16, Lem. 10].

Proposition 15

- (1) $V = (V_K)_{K \leq 0}$ is a basis of the $\langle \underline{s}, \underline{x} \rangle$ -adic topology.
- (2) If $\text{lead}(\underline{s}^\alpha G_k \underline{x}^\beta) \in V_K$ then also $\underline{s}^\alpha G_k \underline{x}^\beta \in V_K$.
- (3) $\pi_H(V_K) = \langle \underline{s} \rangle^{-K} H$.

Proof.

- (1) This follows from (10) and (11).
- (2) Since \underline{g} is a standard basis,

$$\min \deg(\underline{m}) + \min \deg(\underline{x}) \leq \min \deg(\underline{g}) \quad (12)$$

and hence, by (9),

$$\begin{aligned} \deg(\underline{s}D\bar{u}_k) &\leq \max\{\deg(s_j \underline{d}^j \bar{u}_k) \mid 1 \leq j \leq m\} \\ &\leq \max\{\deg(s_j) + \max \deg(\underline{d}^j) + \max \deg(\bar{u}_k) \mid 1 \leq j \leq m\} \\ &\leq \max\{\deg(s_j) + \max \deg(\underline{d}^j) \mid 1 \leq j \leq m\} \\ &\leq \min \deg(\underline{m}) + \min \deg(\underline{x}) \\ &\leq \min \deg(\underline{g}) \leq \deg(g_k) \\ &= \deg \text{lead}(g_k). \end{aligned}$$

Since $\underline{s}^\alpha G_k \underline{x}^\beta = \underline{s}^\alpha (g_k - \underline{s}D\bar{u}_k) \underline{x}^\beta$, this implies that

$$\deg \text{lead}(\underline{s}^\alpha G_k \underline{x}^\beta) = \deg(\underline{s}^\alpha \text{lead}(g_k) \underline{x}^\beta) = \deg(\underline{s}^\alpha G_k \underline{x}^\beta).$$

Hence, the claim follows from (11).

- (3) Let $0 \neq p \in V_K$ and $\underline{s}^\alpha p_\alpha = \text{lead}_{\underline{s}}(p)$ with maximal $|\alpha| < -K$ for fixed $p \bmod \langle \underline{F} \rangle \mathbb{K}[[\underline{s}, \underline{x}]]$. Then, by (9),

$$\begin{aligned} \deg(p_\alpha) &= \deg(\underline{s}^\alpha p_\alpha) - \deg(\underline{s}^\alpha) \\ &< -(K + |\alpha|) \min \deg(\underline{s}) - \min \deg(\underline{x}) + \max \deg(D) \\ &\leq \min \deg(\underline{s}) - \min \deg(\underline{x}) + \max \deg(D) \\ &\leq \min \deg(\underline{m}) \end{aligned}$$

and hence, by (7), $p_\alpha \in \langle \underline{g} \rangle \mathbb{K}[[\underline{x}]]$. By the division theorem, there is a $\bar{q} \in \mathbb{K}[[\underline{x}]]$ with $p_\alpha = \underline{g}\bar{q}$ and $\text{lead}(p_\alpha) \geq \text{lead}(g_j q^j)$ for all j and hence

$$\max \deg(\bar{q}) \leq \deg(p_\alpha) - \min \deg(\underline{g}). \quad (13)$$

Then

$$p_\alpha = \underline{g}\bar{q} = \underline{f}U\bar{q} \equiv \underline{s}DU\bar{q} \bmod \langle \underline{F} \rangle \mathbb{K}[[\underline{s}, \underline{x}]] \quad (14)$$

and hence, by (9), (12), and (13)

$$\begin{aligned} \deg(\underline{s}DU\bar{q}) &\leq \max \deg(\underline{s}D) + \max \deg(U) + \max \deg(\bar{q}) \\ &\leq \max \deg(\underline{s}D) + \max \deg(\bar{q}) \\ &\leq \max \deg(\underline{s}D) - \min \deg(\underline{g}) + \deg(p_\alpha) \\ &\leq \max \deg(\underline{s}D) - \min \deg(\underline{m}) - \min \deg(\underline{x}) + \deg(p_\alpha) \\ &\leq \max\{\deg(s_j) + \max \deg(\underline{d}^j) \mid 1 \leq j \leq m\} \\ &\quad - \min \deg(\underline{m}) - \min \deg(\underline{x}) + \deg(p_\alpha) \\ &\leq \deg(p_\alpha). \end{aligned}$$

Hence, by (14),

$$p' = p - \text{lead}_{\underline{s}}(p) + \underline{s}^{\alpha} \underline{s} D U \bar{q} \equiv p \pmod{\langle \underline{F} \rangle \mathbb{K}[\underline{s}, \underline{x}]}$$

with $\deg(p') \leq \deg(p) < N_K$ and $\text{lead}_{\underline{s}}(p') <_{\underline{s}} \text{lead}_{\underline{s}}(p)$. This contradicts to the maximality of $|\underline{\alpha}|$ and hence $p \in \langle \underline{s} \rangle^{-K} + \langle \underline{F} \rangle \mathbb{K}[\underline{s}, \underline{x}]$. \square

Proposition 15 leads to the following normal form algorithm.

Algorithm 1

```

proc NF( $p \in \mathbb{K}[\underline{s}, \underline{x}], K \leq 0$ )
  if  $p \in \langle \underline{s} \rangle^{-K}$  then  $q := p$ 
  else if  $\deg \text{lead}(p) < N_K$  or  $\text{lead}_{\underline{s}}(p) \in \langle \underline{s} \rangle^{-K}$  then  $q := \text{lead}_{\underline{s}}(p)$ 
  else  $q := 0$ 
   $r := p - q$ 
  if  $r = 0$  then return  $r \in \mathbb{K}[\underline{s}, \underline{x}], \bar{a} \in \mathbb{K}[\underline{s}, \underline{x}]^l, q \in \mathbb{K}[\underline{s}, \underline{x}]$ 
  if  $\text{lead}(r) \in \langle \text{lead}(g) \rangle$  then
     $j := \min\{i \mid \text{lead}(r) \in \langle \text{lead}(g_i) \rangle\}$ 
     $r, \bar{a}, q' := \text{NF}\left(r - \frac{\text{lead}(r)}{\text{lead}(g_j)} g_j - \underline{s} D \left(\frac{\text{lead}(r)}{\text{lead}(g_j)} \bar{u}_j\right), K\right)$ 
     $\bar{a} := \bar{a} + \frac{\text{lead}(r)}{\text{lead}(g_j)} \bar{e}_j$ 
  else
     $r', \bar{a}, q' := \text{NF}(r - \text{lead}(r), K)$ 
     $r := \text{lead}(r) + r'$ 
   $q := q + q'$ 
  return  $r \in \mathbb{K}[\underline{s}, \underline{x}], \bar{a} \in \mathbb{K}[\underline{s}, \underline{x}]^l, q \in \mathbb{K}[\underline{s}, \underline{x}]$ .

```

The input of the algorithm NF is a power series $p \in \mathbb{K}[\underline{s}, \underline{x}]$ and an integer $K \leq 0$, the output is a power series $r \in \mathbb{K}[\underline{s}, \underline{x}]$, a column vector \bar{a} with coefficients in $\mathbb{K}[\underline{s}, \underline{x}]$, and a power series $q \in \mathbb{K}[\underline{s}, \underline{x}]$. We denote the components of NF by

$$(\text{NF}_1(p, K), \text{NF}_2(p, K), \text{NF}_3(p, K)) = (r, \bar{a}, q) = \text{NF}(p, K)$$

for $p \in \mathbb{K}[\underline{s}, \underline{x}]$ and $K \leq 0$.

Example 16 In example 10 using 14, one computes

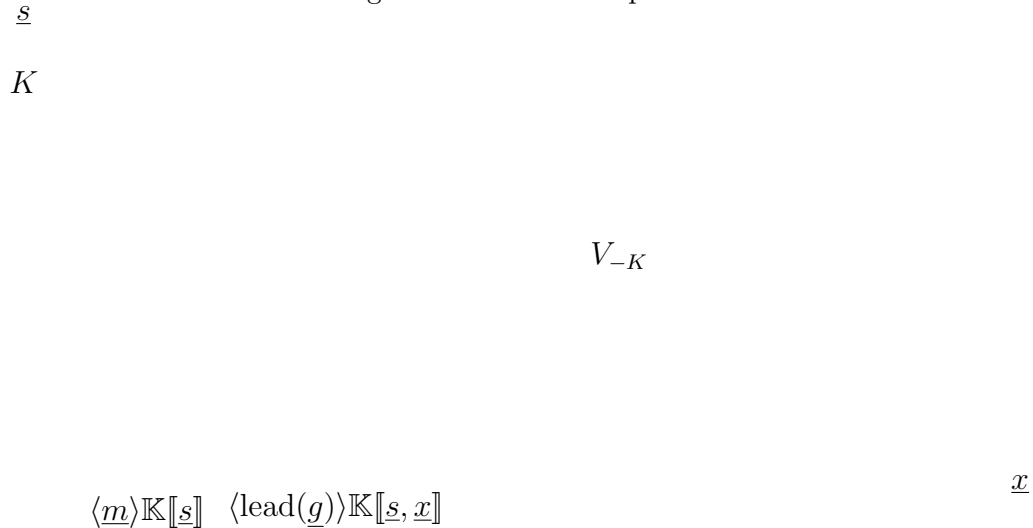
$$\text{NF}_1(\underline{f} \underline{m}, -2) = \underline{m}(A_0 + s A_1)$$

where $A_0, A_1 \in \mathbb{C}^{11 \times 11}$ such that

$$A_0 + sA_1 = \begin{pmatrix} \frac{3}{2}s & 0 & 0 & 0 & -\frac{25}{4}s & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{13}{10}s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{75}{16}s & 0 \\ 0 & 0 & \frac{11}{10}s & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4}s & 0 & 0 \\ 0 & 0 & 0 & \frac{9}{10}s & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{7}{10}s & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{75}{16}s & \frac{13}{10}s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4}s & 0 & 0 & 0 & \frac{11}{10}s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{10}s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{10}s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}s \end{pmatrix}.$$

Figure 1 illustrates a reduction step in NF. The 1-dimensional \mathbb{K} -vector space spanned by a monomial in $\langle \underline{m} \rangle \mathbb{K}[\underline{s}]$, resp. in $\langle \text{lead}(\underline{g}) \rangle \mathbb{K}[\underline{s}, \underline{x}]$, is depicted by a big, resp. small, bullet. The monomial at the tail of the arrow is replaced by a power series with support above the dotted line meeting the head of the arrow. The $\mathbb{K}[\underline{s}]$ -submodule V_K generated by the monomials above the dashed line is invariant with respect to such a reduction step.

Fig. 1. A reduction step in NF



Lemma 17 NF *terminates*.

Proof. For fixed leading exponent $\text{lex}_{\underline{s}}(p)$ with respect to \underline{s} , the leading term $\text{lead}(p)$ is strictly decreasing with weighted degree $\deg \text{lead}(p) \geq N_K$. Since there are only finitely many monomials with fixed $\text{lex}_{\underline{s}}$ and $\deg \geq N_K$, $\text{lex}_{\underline{s}}(p)$ decreases after finitely many steps. Since $\leq = (\leq_{\underline{s}}, \leq_{\underline{x}})$ is a block ordering and $\leq_{\underline{s}}$ is a degree ordering, this implies that $p \in \langle \underline{s} \rangle^{-K}$ after finitely many steps. \square

The following lemma states that $\text{NF}_1(\cdot, L)$ is a reduced normal form modulo V_L with $\text{NF}_1(V_K, L) \subset V_K$ for $L < K \leq 0$.

Lemma 18 *Let $L < K \leq 0$, $p \in V_K$, and $(r, \bar{a}, q) = \text{NF}(p, L)$. Then*

- (1) $p = G\bar{a} + r + q$,
- (2) $a^i \in \mathbb{K}[\underline{s}, \underline{x}]$ with $\text{lead}(G_i a^i) \leq \text{lead}(p)$ for $i = 1, \dots, l$,
- (3) $r \in \langle \underline{m} \rangle \oplus_{L < |\alpha| \leq K} \mathbb{K}\underline{s}^\alpha$ with $\text{lead}(r) \leq \text{lead}(p)$,
- (4) $q \in V_L$.
- (5) If $p \in \langle \underline{m} \rangle \oplus_{L < |\alpha| \leq K} \mathbb{K}\underline{s}^\alpha$ then $(r, \bar{a}, q) = (p, \bar{0}, 0)$.
- (6) If $p \equiv p' \pmod{V_L}$ then $r = r'$.

Proof. By Proposition 15.2, NF preserves the condition $p \in V_K$. Hence, the claim follows immediately from the definition of NF. \square

By Proposition 15.1, V is a basis of the $\langle \underline{s}, \underline{x} \rangle$ -adic topology and, by Lemma 18, $\text{NF}_1(V_K, L) \subset V_K$ and $\text{NF}_3(V_K, L) \subset V_L$ for $L < K \leq 0$. Since $\mathbb{K}[[\underline{s}, \underline{x}]]$ is complete with respect to the $\langle \underline{s}, \underline{x} \rangle$ -adic topology,

$$\mathbb{K}[[\underline{s}, \underline{x}]] = \varprojlim_{\kappa} (\mathbb{K}[[\underline{s}, \underline{x}]]/V_K)$$

and hence NF induces a reduced normal form on $\mathbb{K}[[\underline{s}, \underline{x}]]$ as follows.

Definition 19 *Let $K = (K_i)_{i \geq 0}$ be a strictly decreasing sequence and*

$$\text{NF}(p) = (\text{NF}_1(p), \text{NF}_2(p)) = \left(\sum_{i \geq 0} r_i, \sum_{i \geq 0} \bar{a}_i \right)$$

for $p \in \mathbb{K}[[\underline{s}, \underline{x}]]$ where $p_0 = p$ and $r_i, \bar{a}_i, p_{i+1} = \text{NF}(p_i, K_i)$ for $i \geq 0$.

Note that NF depends on the choice of the sequence K .

Lemma 20 *Let $(r, \bar{a}) = \text{NF}(p)$. Then*

- (1) $p = G\bar{a} + r$,
- (2) $a^i \in \mathbb{K}[[\underline{s}, \underline{x}]]$ with $\text{lead}(G_i a^i) \leq \text{lead}(p)$ for $i = 1, \dots, l$,

- (3) $r \in \langle \underline{m} \rangle \mathbb{K}[[\underline{s}]]$ with $\text{lead}(r) \leq \text{lead}(p)$.
(4) If $p \in \langle \underline{m} \rangle \mathbb{K}[[\underline{s}]]$ then $(r, \bar{a}) = (p, \bar{0})$.

Proof. This follows immediately from Proposition 15.1 and Lemma 18. \square

The following proposition describes NF_1 as a map of $\mathbb{C}[[\underline{s}]]$ -modules.

Proposition 21 NF_1 is a $\mathbb{K}[[\underline{s}]]$ -linear map

$$\begin{array}{ccc} \mathbb{K}[[\underline{s}, \underline{x}]] & \xrightarrow{\text{NF}_1} & \langle \underline{m} \rangle \mathbb{K}[[\underline{s}]] \\ & \searrow \pi_H & \swarrow \pi_H \\ & H & \end{array}$$

with $\text{NF}_1^2 = \text{NF}_1$. In particular, if H is a free $\mathbb{K}[[\underline{s}]]$ -module then NF_1 induces the $\mathbb{K}[[\underline{s}]]$ -section

$$\mathbb{K}[[\underline{s}, \underline{x}]] \xleftarrow{\text{NF}_1} H \xrightarrow{\pi_H} \mathbb{K}[[\underline{s}]]$$

of the canonical projection π_H with image $\text{NF}_1(H) = \langle \underline{m} \rangle \mathbb{K}[[\underline{s}]]$. This means that NF_1 is the \underline{m} -basis representation.

Proof. By definition of NF , NF_1 is $\mathbb{K}[[\underline{s}]]$ -linear. By Lemma 20, NF_1 is a map over H . By Lemma 20.4, NF_1 is the identity on its image $\langle \underline{m} \rangle \mathbb{K}[[\underline{s}]]$. \square

If H is not a free $\mathbb{K}[[\underline{s}]]$ -module then its relations can be computed as follows. By Proposition 21,

$$\begin{aligned} H &\cong_{\mathbb{K}[[\underline{s}]]} \langle \underline{m} \rangle \mathbb{K}[[\underline{s}]] / \left(\langle \underline{m} \rangle \mathbb{K}[[\underline{s}]] \cap \langle \underline{F} \rangle \mathbb{K}[[\underline{s}, \underline{x}]] \right) \\ &= \langle \underline{m} \rangle \mathbb{K}[[\underline{s}]] / \text{NF}_1 \left(\langle \underline{F} \rangle \mathbb{K}[[\underline{s}, \underline{x}]] \right) \\ &= \langle \underline{m} \rangle \mathbb{K}[[\underline{s}]] / \left\langle \text{NF}_1 \left(\underline{F} \underline{x}^{\text{Nn}} \right) \right\rangle \mathbb{K}[[\underline{s}]] \end{aligned}$$

and, in particular, by Lemma 18.3,

$$H / \langle \underline{s} \rangle^{-K} H \cong_{\mathbb{K}[[\underline{s}]]} \langle \underline{m} \rangle \mathbb{K}[[\underline{s}]] / \left(\left\langle \text{NF}_1 \left(\underline{F} \underline{x}^{\text{Nn}} \setminus V_K, K \right) \right\rangle \mathbb{K}[[\underline{s}]] + \langle \underline{m} \rangle \langle \underline{s} \rangle^{-K} \mathbb{K}[[\underline{s}]] \right)$$

where $\underline{F} \underline{x}^{\text{Nn}} \setminus V_K$ is a finite subset of $\mathbb{K}[[\underline{s}, \underline{x}]]$.

Finally, we return to our starting point. Let $\mathbb{K} = \mathbb{C}$ and $\underline{F} = \partial(f) - s\partial$ as in Remark 8. Then, by Proposition 7, $H \cong_{\mathbb{C}[[\underline{s}]]} \widehat{H}''$ is the formal Brieskorn lattice and, by Proposition 12, H is a free $\mathbb{C}[[\underline{s}]]$ -module of rank μ . We define the matrix $A^m \in \mathbb{C}[[\underline{s}]]^{\mu \times \mu}$ by

$$\underline{m} A^m = \underline{t} m.$$

Then, by (3), $A^m + s^2\partial_s$ is the \underline{m} -basis representation of t . This means that there is a commutative diagram

$$\begin{array}{ccc} \widehat{H}'' & \xrightarrow{t} & \widehat{H}'' \\ \underline{m} \uparrow \sim & & \underline{m} \uparrow \sim \\ \mathbb{C}[[s]]^\mu & \xrightarrow{A^m + s^2\partial_s} & \mathbb{C}[[s]]^\mu. \end{array}$$

By (1) and proposition 21,

$$\underline{m}A^m = \text{NF}_1(fm).$$

Example 22 *In example 10 using 16,*

$$A^m \equiv A_0 + sA_1 \pmod{s^2\mathbb{C}[[s]]^{11 \times 11}}.$$

By [16, Sec. 8], the non-diagonal terms of A_1 and the terms of A^m in $s^2\mathbb{C}[[s]]^{11 \times 11}$ can be eliminated by transforming \underline{m} to a good $\mathbb{C}[[s]]$ -basis of \widehat{H}'' . Then

$$A^m = A_0 + sA_1$$

and A_0 and A_1 represent M. Saito's endomorphisms [7].

All the algorithms in [13–16] require the computation of the matrix A^m for a $\mathbb{C}[[s]]$ -basis \underline{m} of \widehat{H}'' .

4 Examples and timings

Algorithm 1 for the case of Remark 8 and the algorithms in [15,16,30] are implemented in the SINGULAR [19] library `gmssing.lib` [18]. We use this implementation on a PENTIUM III M 1 GHz machine with 512 MB of memory plus 1 GB of swap memory. For several polynomials $f \in \mathbb{C}[\underline{x}]$ with isolated critical point at the origin, we compute

- (1) the local Bernstein-Sato polynomial,
- (2) the spectral pairs, and
- (3) M. Saito's endomorphisms A_0 and A_1 .

For $i = 1, 2, 3$, we denote by t_i the corresponding computation time in seconds and by K_i the maximal K occurring in NF during the computation. By t_K we denote the time in seconds needed to compute $A^m \pmod{s^K\mathbb{C}[[s]]^{\mu \times \mu}}$. All computation times are rounded off.

The local Bernstein-Sato polynomial at the origin for the examples in [11, Tab. 1] can be computed, each in less than one second. Table 1 shows the

timings for the examples in [11, Tab. 2]. By [15], it suffices to compute $A^m \bmod s^{K_0} \mathbb{C}[[s]]^{\mu \times \mu}$ where $K_0 = 2(\mu + n - 1)$ in order to compute all of the above invariants. Table 2 shows the results for the examples in [13, Tab. 2] and that this a priori bound is useless in practice.

Table 1

Local Bernstein-Sato polynomial b for $f = x^{n_1} + y^{n_2} + z^{n_3} + x^{m_1} y^{m_2} z^{m_3}$

\underline{n}	6, 6, 6	7, 7, 7	7, 7, 7	9, 9, 9	6, 6, 7	6, 6, 7	6, 6, 7	6, 7, 7
\underline{m}	4, 4, 4	2, 2, 2	2, 2, 3	3, 3, 3	2, 2, 2	3, 3, 3	4, 4, 4	2, 2, 2
μ	125	167	216	512	138	150	150	152
$\deg(b)$	13	28	17	23	60	48	52	62
K_1	3	4	2	2	4	3	3	4
t_1	1	5	2	50	6	1	2	6

Table 2

Spectral pairs and M. Saito's endomorphisms for [13, Tab. 2]

	f	μ	K_2	t_{K_2}	t_2	K_3	t_{K_3}	t_3	K_0	t_{K_0}
$Z_{1,1}$	$x^3y + x^2y^3 + y^8$	16	2	0	0	4	0	10	34	180
$W_{1,1}$	$x^4 + x^2y^3 + y^7$	16	2	0	0	4	0	2	34	39
$W_{1,1}^\#$	$x^4 + 2x^2y^3 + xy^5 + y^6$	16	2	0	1	4	1	54	34	495
$Q_{2,1}$	$x^3 + yz^2 + x^2y^2 + y^7$	15	2	0	1	4	0	4	34	58
$Q_{2,2}$	$x^3 + yz^2 + x^2y^2 + y^8$	16	2	0	0	4	0	1	36	20
$S_{1,1}$	$x^2z + yz^2 + x^2y^2 + y^6$	15	2	0	0	4	0	1	34	43
$S_{1,2}$	$x^2z + yz^2 + x^2y^2 + y^7$	16	2	0	0	4	0	0	36	27
$S_{1,1}^\#$	$x^2z + yz^2 + y^3z + xy^4$	15	2	0	1	4	0	18	34	687
$S_{1,2}^\#$	$x^2z + yz^2 + y^3z + x^2y^3$	16	2	0	1	4	0	2	36	252
$U_{1,1}$	$x^3 + xz^2 + xy^3 + y^2z^2$	15	2	0	1	4	0	2	34	66
$U_{1,2}$	$x^3 + xz^2 + xy^3 + y^4z$	16	2	0	0	4	0	0	36	12
$V_{1,1}$	$x^2y + z^2y^2 + z^4 + y^5$	16	2	0	0	6	0	2	36	12
$V_{1,1}^\#$	$x^2y + y^4 + xz^3 + yz^3$	16	2	0	1	6	0	1	36	10

By Remark 8, (6) and (8), the coefficient of s^K in the \underline{m} -basis representation in \widehat{H}'' is defined by a division by the ideal $\langle \partial(f) \rangle \mathbb{C}[[x]]$ where the output of the division for s^K defines the input for the division for s^{K+1} . Therefore, the complexity of the data and the computation time increases rapidly with K . In [13, Sec. 10.2], we compute such a division by a sequence of weak normal form computations. Table 3 shows the time needed to compute $A^m \bmod s^K \mathbb{C}[[s]]^{\mu \times \mu}$ for example 10 and increasing K using this method and the

SINGULAR command `division`. The computation fails in degree $K = 8$ after more than one hour due to lack of memory. In the algorithm NF, the above sequence of full divisions is replaced by a sequence of partial divisions. Table 4 shows the time needed to compute the same result using the algorithm NF. The situation is similar for other examples.

Table 3

A^m for $f = x^5 + x^2y^2 + y^5$ using `division`

K	4	5	6	7	8
t_K	0	1	10	283	∞

Table 4

A^m for $f = x^5 + x^2y^2 + y^5$ using NF

K	20	40	60	80	100	120	140	160	180	200
t_K	1	2	8	18	35	64	106	165	247	354

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