

1 **DERIVATIONS OF NEGATIVE DEGREE ON QUASIHOMOGENEOUS**
2 **ISOLATED COMPLETE INTERSECTION SINGULARITIES**

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ABSTRACT. J. Wahl conjectured that every quasihomogeneous isolated normal singularity admits a positive grading for which there are no derivations of negative weighted degree. We confirm his conjecture for quasihomogeneous isolated complete intersection singularities of either order at least 3 or embedding dimension at most 5. For each embedding dimension larger than 5 (and each dimension larger than 3), we give a counter-example to Wahl's conjecture.

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11 1. INTRODUCTION

12 By a singularity we mean a quotient A of a convergent power series ring over a valued
13 field K of characteristic zero (see §2). We use the acronym *negative derivation* for a
14 derivation of negative weighted degree on a quasihomogeneous singularity. The question
15 of existence of such negative derivations has important consequences in rational homotopy
16 theory (see [Mei82, Thm. A]) and in deformation theory (see [Wah82, Thm. 3.8]).

17 By a result of Kantor [Kan79], quasihomogeneous curve and hypersurface singularities
18 do not admit any negative derivations. J. Wahl [Wah82, Thm. 2.4, Prop. 2.8] reached the
19 same conclusion in (the much deeper) case of quasihomogeneous normal surface singular-
20 ities. Motivated by his cohomological characterization of projective space in [Wah83a],
21 he formulates the following conjecture in [Wah83b, Conj. 1.4].

22 **Conjecture** (Wahl). *Let R be a normal graded ring, with isolated singularity. Then there*
23 *is a normal graded \bar{R} , with $\hat{R} \cong \hat{\bar{R}}$, so that \bar{R} has no derivations of negative weight.*

24 In case R is a graded normal locally complete intersection with isolated singularity,
25 \hat{R} becomes a quasihomogeneous normal isolated complete intersection singularity (ICIS)
26 and Wahl's conjecture can be rephrased as follows (see Lemma 2.1 and Remark 2.3).

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27 **Conjecture** (Wahl, ICIS case). *Any quasihomogeneous normal ICIS has no negative*
 28 *derivations with respect to some positive grading.*

29 For quasihomogeneous normal ICIS, there is an explicit description of all derivations
 30 due to Kersken [Ker84]. Based on this description, we prove our main

31 **Theorem 1.1.** *For any quasihomogeneous normal ICIS of order at least 3 there are no*
 32 *negative derivations with respect to any positive grading.*

33 *Proof.* This follows from Corollary 3.4 and Proposition 3.8. □

34 Our investigations lead to a family of counter-examples to Wahl's Conjecture. In
 35 order to describe it, we first fix our notation. A quasihomogeneous singularity can be
 36 represented as

$$(1.1) \quad A = P/\mathfrak{a}, \quad \mathfrak{a} = \langle g_1, \dots, g_t \rangle \subseteq K\langle\langle x_1, \dots, x_n \rangle\rangle =: P$$

where g_1, \dots, g_t are homogeneous polynomials of degree $p_i := \deg(g_i)$ with respect to
 weights $w_1, \dots, w_n \in \mathbb{Z}_+$ on the variables x_1, \dots, x_n (see §2). We order these weights and
 degrees decreasingly as

$$(1.2) \quad \begin{aligned} w_1 &\geq \dots \geq w_n > 0, \\ p_1 &\geq \dots \geq p_t. \end{aligned}$$

Example 1.2. Let $n \geq 6$ and pick $c_7, \dots, c_n \in K \setminus \{1\}$ pairwise different such that $c_i^9 + 1 \neq 0$
 for all i . Assigning weights $8, 8, 5, 2, \dots, 2$ to the variables x_1, \dots, x_n , the equations

$$(1.3) \quad \begin{aligned} g_1 &:= x_1x_4 + x_2x_5 + x_3^2 - x_4^5 + \sum_{i=7}^n x_i^5 \\ g_2 &:= x_1x_5 + x_2x_6 + x_3^2 + x_6^5 + \sum_{i=7}^n c_i x_i^5 \end{aligned}$$

37 of degree 10 define a quasihomogeneous complete intersection A as in (1.1) with isolated
 38 singularity. On A there is a derivation

$$(1.4) \quad \eta := \begin{vmatrix} \partial_1 & \partial_2 & \partial_3 \\ x_4 & x_5 & 2x_3 \\ x_5 & x_6 & 2x_3 \end{vmatrix} = 2x_3(x_5 - x_6)\partial_1 - 2x_3(x_4 - x_5)\partial_2 + (x_4x_6 - x_5^2)\partial_3$$

39 of degree -1 . We work out the details of this example in §5.

40 We show that Example 1.2 gives a counter-example to the ICIS case of Wahl's conjec-
 41 ture of minimal embedding dimension $n = 6$.

42 **Theorem 1.3.** *Exactly up to embedding dimension 5, all quasihomogeneous ICIS have*
 43 *no negative derivations with respect to some positive grading.*

44 *Proof.* This follows from Kantor [Kan79], [Wah82, Thm. 2.4, Prop. 2.8], Proposition 4.2,
 45 Example 1.2 and Corollary 3.4. □

46 As a consequence of our arguments we obtain a simple special case of the following
 47 conjecture due to S. Halperin.

48 **Conjecture** (Halperin). *On any graded zero-dimensional complete intersection there are*
 49 *no negative derivations.*

50 The following result bounds the degree of negative derivations (see also [Ale91, Prop.]).
 51 The bound does not require a complete intersection hypothesis and it is independent of
 52 further hypotheses as for instance in [Hau02, Thm. 2].

53 **Proposition 1.4.** *For any quasihomogeneous zero-dimensional singularity A as in (1.1)*
54 *there are no derivations of degree strictly less than $p_n - p_1$. In particular, Halperin's*
55 *conjecture holds true if $p_1 = p_n$.*

56 *Proof.* As A is assumed to be zero-dimensional, condition $\mathfrak{A}(k)$ on page 4 must hold true
57 for all $k = 1, \dots, n$. Then the claim follows from Remark 3.6 and Lemma 3.7. \square

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61 2. GRADED ANALYTIC ALGEBRAS

62 Consider a (local) analytic algebra $A = (A, \mathfrak{m}_A)$ over a (possibly trivially) valued field K
63 of characteristic zero. We assume in addition that A is non-regular and can be represented
64 as a quotient $A = P/\mathfrak{a}$ of a convergent power series ring $P := K\langle\langle x_1, \dots, x_n \rangle\rangle \supseteq \mathfrak{a}$. In
65 the sequel such an A will be referred to as a *singularity*. We choose n minimal such that
66 $n = \text{embdim } A$ and set $d := \dim A$.

67 A K_+ -grading on A is given by a *diagonalizable derivation* $\chi \in \text{Der}_K A =: \Theta_A$
68 which means that \mathfrak{m}_A is generated by eigenvectors x_1, \dots, x_n (see [SW73, (2.2),(2.3)]).
69 Such a derivation is also called an *Euler derivation*. We refer to w_1, \dots, w_n defined by
70 $w_i := \chi(x_i)/x_i$ as the *eigenvalues of χ* . More generally, we call χ -eigenvectors $f \in A$
71 (χ) -homogeneous and define their (χ) -degree to be the corresponding eigenvalue denoted
72 by $\deg(f) := \chi(f)/f \in k$. We denote by A_a the K -vector space of all such eigenvector
73 $f \in A$ with $\deg(f) = a$. This defines a K -subalgebra

$$(2.1) \quad \bar{A} := \bigoplus_{a \in K} A_a \subset A \subset \hat{A}.$$

74 The derivation $\chi \in \Theta_A$ lifts to $\chi \in \Theta_P := \text{Der}_K P$ (see [SW73, (2.1)]). In particular, P
75 is K_+ -graded and $\mathfrak{a} \subseteq P$ is a χ -invariant ideal and hence (χ) -homogeneous (see [SW73,
76 (2.4)]). Pick homogeneous $g_1, \dots, g_t \in \mathfrak{a}$ inducing a K -vector space basis of $\mathfrak{a}/\mathfrak{m}_A \mathfrak{a}$. Then
77 $\mathfrak{a} = \langle g_1, \dots, g_t \rangle$ by Nakayama's Lemma. We set $p_i := \deg(g_i)$ ordered as in (1.2). To
78 summarize, we can write A as in (1.1).

79 A K_+ -grading is called a positive grading if $w_i \in \mathbb{Z}_+$ for all $i = 1, \dots, n$ (see [SW73,
80 §3, Def.]). We call A *quasihomogeneous* if it admits a positive grading. In this case, we
81 shall always normalize χ to make the w_i coprime and order the variables according to
82 (1.2). Positivity of weights enforces $g_i \in \bar{P} = K[x_1, \dots, x_n]$ and that

$$(2.2) \quad \bar{A} = \bigoplus_{i \geq 0} A_i = \bar{P}/\bar{\mathfrak{a}}, \quad \bar{\mathfrak{a}} = \langle g_1, \dots, g_t \rangle \subseteq K[x_1, \dots, x_n] = \bar{P},$$

83 is a (positively) graded-local k -algebra with completion

$$(2.3) \quad \hat{\bar{A}} = \hat{A}$$

84 and graded maximal ideal $\mathfrak{m}_{\bar{A}} = \bar{\mathfrak{m}}_A := \bigoplus_{i > 0} A_i$. The preceding discussion enables us to
85 reformulate Wahl's Conjecture in the language of Scheja and Wiebe as follows.

86 **Lemma 2.1.** *The following supplementary structures on a singularity A are equivalent:*

- 87 (1) *an Euler derivation χ on A with positive eigenvalues,*
- 88 (2) *a positive grading on A ,*
- 89 (3) *a positive grading on \hat{A} ,*
- 90 (4) *a (positively) graded K -algebra \bar{A} such that $\hat{\bar{A}} = \hat{A}$.*

91 *Proof.* The equivalences of (1), (2), and (3) are due to Scheja and Wiebe (see [SW73,
 92 (2.2),(2.3)] and [SW77, (1.6)]). For the equivalence with (4), note that the obvious Euler
 93 derivation on a graded K -algebra \bar{A} lifts to an Euler derivation on the completion $\hat{\bar{A}} = \hat{A}$.
 94 The converse follows from (2.1), (2.2) and (2.3). \square

95 Let us assume now that A is an isolated complete intersection singularity (ICIS). We
 96 may then take g_1, \dots, g_t to be a regular sequence and $d + t = n$. The isolated singularity
 97 hypothesis can be expressed in terms of the Jacobian ideal

$$(2.4) \quad J_A := \left\langle \left| \frac{\partial g}{\partial x_\nu} \right| \mid |\nu| = t \right\rangle \trianglelefteq A$$

98 of A as follows.

99 **Proposition 2.2.** *A complete intersection singularity A is isolated if and only if J_A is*
 100 *\mathfrak{m}_A -primary. An analogous statement holds for \bar{A} .*

101 *Proof.* We denote by $\Omega_{A/k}^1$ the universally finite module of differentials of A over k . By
 102 the standard sequence

$$\mathfrak{a}/\mathfrak{a}^2 \longrightarrow A \otimes_P \Omega_{P/k}^1 \longrightarrow \Omega_{A/k}^1 \longrightarrow 0,$$

103 the Jacobian ideal J_A is the d th Fitting ideal $F_A^d \Omega_{A/k}^1$. By [SS72, (6.4),(6.9)], reducedness
 104 of A is equivalent to $\text{rk} \Omega_{A/k}^1 = d$ and $A_{\mathfrak{p}}$ is regular if and only if $\Omega_{A_{\mathfrak{p}}/k}^1$ is free. Hence,
 105 $A_{\mathfrak{p}}$ being regular is equivalent to $\mathfrak{p} \not\supseteq F_A^0 \Omega_{A/k}^1 = J_A$ by [BH93, Lem. 1.4.9]. In particular,
 106 A having an isolated singularity means exactly that A/J_A is supported at \mathfrak{m}_A and hence
 107 that J_A is \mathfrak{m}_A -primary as claimed. The analogous statement for \bar{A} is proved similarly. \square

108 *Remark 2.3.* Let A be a quasihomogeneous singularity. By (2.2),

$$(2.5) \quad J_{\bar{A}} := \bar{J}_A = \left\langle \left| \frac{\partial g}{\partial x_\nu} \right| \mid |\nu| = t \right\rangle \trianglelefteq \bar{A}$$

109 is the Jacobian ideal of \bar{A} defined analogous to (2.4). By (2.3), A is a complete intersection
 110 if and only if \bar{A} is locally a complete intersection (see [BH93, Def. 2.3.1, Ex. 2.3.21.(c)]).
 111 By Proposition 2.2, A is an ICIS if and only if J_A is \mathfrak{m}_A -primary. This is equivalent to $J_{\bar{A}}$
 112 being $\mathfrak{m}_{\bar{A}}$ -primary. The latter is then equivalent to \bar{A} being locally a complete intersection
 113 with isolated singularity by (2.5) and Proposition 2.2. Complete intersections are Cohen–
 114 Macaulay and hence (S_2) so normality is equivalent to (R_1) by Serre’s Criterion (see
 115 [BH93, §2.3, Thm. 2.2.22]). Since $d = \dim A = \dim \bar{A}$ by (2.3) (see [BH93, Cor. 2.1.8]),
 116 normality for both A and \bar{A} reduces to $d \geq 2$.

117 Scheja and Wiebe [SW77, (3.1)] (see also [Sai71, Satz 1.3]) proved that any K_+ -graded
 118 ICIS is quasihomogeneous unless $t = 1$ and $g_1 \notin \mathfrak{m}_P^3$. Their starting point (see [SW77,
 119 (2.5)] and [Sai71, Lem. 1.5]) is that A being an ICIS implies, by Proposition 2.2, that for
 120 each $k = 1, \dots, n$ one of the following two conditions must hold true.

- 121 $\mathfrak{A}(k)$ For some $m \geq 2$ and $1 \leq j \leq t$, the monomial x_k^m occurs in g_j .
- 122 $\mathfrak{B}(k)$ For some pairwise different $1 \leq \nu_1, \dots, \nu_t \leq n$, each g_j contains a monomial $x_k^{m_j} x_{\nu_j}$
- 123 for some $m_j \geq 1$.

124 The following result gives numerical constraints for A to be a quasihomogeneous ICIS.

125 **Lemma 2.4.** *If A is a quasihomogeneous ICIS then*

$$(2.6) \quad p_1 + \dots + p_j \geq w_1 + \dots + w_j + jw_n$$

126 for all $j = 1, \dots, t$.

127 *Proof.* We proceed by induction on j . Assume that (2.6) holds for $j - 1$ but not for j ,
 128 that is, $p_1 + \dots + p_j \leq w_1 + \dots + w_j + (j - 1)w_n$ using (1.2). Then $p_j \leq w_j$ and hence
 129 $g_i = g_i(x_{j+1}, \dots, x_n)$ for all $i = j, \dots, t$. Then J_A maps to zero in

$$A/\langle x_{j+1}, \dots, x_n \rangle = K\langle\langle x_1, \dots, x_j \rangle\rangle/\langle g_1, \dots, g_{j-1} \rangle$$

130 and hence J_A cannot be \mathfrak{m}_A -primary as required by Proposition 2.2. \square

131

3. NEGATIVE DERIVATIONS

132 Let A be a quasihomogeneous singularity as in §2. The target of our investigations
 133 is the positively graded A -module $\Theta_A = \text{Der}_K A$ of K -linear derivations on A . More
 134 precisely, we are concerned with the question whether its negative part

$$\Theta_{A, < 0} = \Theta_{\bar{A}, < 0} = \bigoplus_{i < 0} \Theta_{A, i}$$

135 is trivial. A priori this condition depends on the choice of a grading. In Proposition 3.1
 136 below, we shall prove the independence of this choice for a general singularity under a
 137 strong hypothesis satisfied in the ICIS case (see Corollary 3.4). To this end, we write (see
 138 [SW73, (2.1)])

$$(3.1) \quad \Theta_A = \Theta_{\mathfrak{a} \subset P} / \mathfrak{a} \Theta_P$$

139 as a quotient of a (k, P) -Lie algebra

$$\Theta_{\mathfrak{a} \subset P} := \{ \delta \in \Theta_P \mid \delta \mathfrak{a} \subset \mathfrak{a} \} \supseteq \mathfrak{a} \Theta_P$$

140 of logarithmic derivations along \mathfrak{a} by the (k, P) -Lie ideal $\mathfrak{a} \Theta_P$.

Proposition 3.1. *Let A be a quasihomogeneous singularity with positive grading given by χ and assume that*

$$(3.2) \quad \Theta_{\mathfrak{a} \subset P} = P\chi + \Theta'_P + \mathfrak{a} \Theta_P,$$

$$(3.3) \quad \text{for some } \Theta'_P \subset \mathfrak{m}_P^2 \Theta_P.$$

141 *Then the condition $\Theta_{A, < 0} = 0$ and the p_1, \dots, p_t in (1.2) are independent of the chosen*
 142 *positive grading.*

143 *Proof.* Consider a second positive grading with corresponding Euler derivation χ' (see
 144 Lemma 2.1). By (3.1) and (3.2), any $\delta \in \Theta_A$ lifts to an element of $\Theta_{\mathfrak{a} \subset P}$ of the form

$$(3.4) \quad \delta = c\chi + \delta_+, \quad \delta_+ = a\chi + \eta, \quad c \in K, \quad a \in \mathfrak{m}_P, \quad \eta \in \Theta'_P,$$

145 denoted by the same symbol. By (3.3) and the Leibniz rule,

$$(3.5) \quad \chi \mathfrak{m}_P^k \subset \mathfrak{m}_P^k, \quad \delta_+ \mathfrak{m}_P^k \subset \mathfrak{m}_P^{k+1}$$

146 for all $k \geq 1$. Specializing to $\delta = \chi$, this implies that $\chi_+ = 0$ and $\chi' = c\chi$ on $\mathfrak{m}_A / \mathfrak{m}_A^2 =$
 147 $\mathfrak{m}_P / \mathfrak{m}_P^2$ and hence $c = 1$ by the definition of a positive grading and our normalization of
 148 weights.

149 Using (3.1), we equip Θ_A with the decreasing \mathfrak{m}_P -adic filtration F^\bullet induced from Θ_P
 150 which is defined as follows

$$F^k \Theta_A = (\Theta_{\mathfrak{a} \subset P} \cap \mathfrak{m}_P^k \Theta_P) / (\mathfrak{a} \Theta_P \cap \mathfrak{m}_P^k \Theta_P).$$

151 Due to (3.3), (3.4) and (3.5) this is a filtration by (k, P) -Lie ideals and

$$\delta_+ F^k \Theta_A \subset F^{k+1} \Theta_A$$

152 for the adjoint action of δ_+ . Therefore, for any $k \geq 1$, the adjoint action of $\chi' = \chi + \chi_+$
 153 on the truncation

$$F^{\leq k} \Theta_A := \Theta_A / F^{k+1} \Theta_A$$

154 is triangularizable with semisimple part equal to that of χ . Thus, χ' and χ have the
 155 same eigenvalues on $F^{\leq k} \Theta_A$ for any $k \geq 1$. The first claim then follows by choosing k
 156 sufficiently large. A similar argument yields the second claim. \square

157 For a Gorenstein singularity A , there is a natural way to produce elements of Θ_A . The
 158 A -submodule $\Theta'_A \subset \Theta_A$ of *trivial derivations* is by definition the image of the inclusion

$$(3.6) \quad \Omega_{A/K}^{d-1} \hookrightarrow \omega_{A/K}^{d-1} = \text{Hom}_A(\Omega_{A/K}^1, \omega_{A/K}^d) = \Theta_A \otimes_A \omega_{A/K}^d \cong \Theta_A.$$

159 We return to the case of an ICIS singularity A . For $1 \leq \nu_0 < \dots < \nu_t \leq n$ with
 160 complementary indices $1 \leq \mu_1 < \dots < \mu_{d-1} \leq n$, the lift to P of the image of $dx_{\mu_1} \wedge \dots \wedge$
 161 $dx_{\mu_{d-1}}$ can be written (up to sign) explicitly as

$$(3.7) \quad \delta_\nu := \begin{vmatrix} \partial_{\nu_0} & \cdots & \partial_{\nu_t} \\ \partial_{\nu_0} g_1 & \cdots & \partial_{\nu_t} g_1 \\ \vdots & & \vdots \\ \partial_{\nu_0} g_t & \cdots & \partial_{\nu_t} g_t \end{vmatrix}.$$

Note that

$$(3.8) \quad \deg \delta_\nu = p_1 + \dots + p_t - w_{\nu_0} - \dots - w_{\nu_t},$$

$$(3.9) \quad \delta_\nu g_j = 0$$

162 for all $j = 1, \dots, t$ and ν . Consider the P -module

$$(3.10) \quad \Theta'_P := \langle \delta_\nu \mid 1 \leq \nu_0 < \dots < \nu_t \leq n \rangle_P \subset \Theta_P.$$

163 The key to our investigations is the following result due to Kersken [Ker84, (5.2)]. From
 164 now on we assume in addition that A is quasihomogeneous and normal, that is, $\dim A \geq 2$.

165 **Theorem 3.2** (Kersken). *Let A be a quasihomogeneous normal ICIS. Then the module*
 166 *Θ_A of K -linear derivations on A is generated by the Euler derivation χ and the trivial*
 167 *derivations Θ'_A .*

168 Although Kersken only states that Θ'_A is minimally generated by the δ_ν in (3.7), his
 169 arguments show that together with χ they form a minimal set of generators of Θ_A . We
 170 denote by $\mu(-)$ the minimal number of generators.

171 **Corollary 3.3.** *Let A be quasihomogeneous normal ICIS. Then Θ_A is minimally gener-*
 172 *ated by the Euler derivation χ and the trivial derivations δ_ν in (3.7). In particular,*

$$\mu(\Theta_A) = \binom{n}{t+1} + 1.$$

173 *Proof.* Since the case $d = 2$ is covered by [Wah87, Prop. 1.12], we may assume that $d \geq 3$.
 174 In this case, the inclusion (3.6) fits into the following commutative diagram with exact

175 rows and columns (see [Ker84, Proof of (4.8)] or [Wah87, Prop. 1.7]).

$$(3.11) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H_{\mathfrak{m}_A}^1(\Omega_{A/K}^d) & \xrightarrow[\cong]{\chi} & H_{\mathfrak{m}_A}^1(\Omega_{A/K}^{d-1}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \omega_{A/K}^d & \xrightarrow{\chi} & \omega_{A/K}^{d-1} & \xrightarrow{\chi} & \omega_{A/K}^{d-2} \\ & & \uparrow & & \uparrow & & \uparrow \cong \\ 0 & \longrightarrow & \Omega_{A/K}^d & \xrightarrow{\chi} & \Omega_{A/K}^{d-1} & \xrightarrow{\chi} & \Omega_{A/K}^{d-2} \\ & & & & \uparrow & & \uparrow \\ & & & & 0 & & 0 \end{array}$$

176 It follows that

$$\chi(\omega_{A/K}^{d-1}) \cong \chi(\Omega_{A/K}^{d-1}) \cong \Omega_{A/K}^{d-1} / \chi(\Omega_{A/K}^d)$$

177 where $\chi(\Omega_{A/K}^d) \subset \mathfrak{m}_A \Omega_{A/K}^{d-1}$ and hence

$$\mu(\chi(\Omega_{A/K}^{d-1})) = \mu(\Omega_{A/K}^{d-1}) = \mu(\Theta'_A).$$

178 Now the middle row of (3.11) yields an exact sequence

$$0 \longrightarrow A \xrightarrow{\chi} \Theta_A \longrightarrow \chi(\omega_{A/K}^{d-1}) \otimes (\omega_{A/K}^d)^{-1} \longrightarrow 0$$

179 Since $\chi \notin \mathfrak{m}_A \Theta_A$, the claim follows. \square

180 Note that Θ'_p in (3.10) satisfies (3.3) due to (3.7) unless $t = 1$ and $g_1 \notin \mathfrak{m}_p^3$. As a
 181 consequence of Proposition 3.1 and Theorem 3.2 we therefore obtain the following result.
 182 It is crucial for Example 1.2 to be a counter-example to Wahl's Conjecture.

183 **Corollary 3.4.** *Let A be a quasihomogeneous normal ICIS. Unless $t = 1$ and $g_1 \notin \mathfrak{m}_p^3$,
 184 the condition $\Theta_{A,<0} = 0$ and the p_1, \dots, p_t in (1.2) are independent of the choice of a
 185 positive grading. \square*

186 We shall now derive numerical constraints for minimal negative trivial derivations. To
 187 this end, suppose that $0 \neq \eta \in \Theta_{A,<0}$. For degree reasons (see (1.2)), η can be written as
 188

$$(3.12) \quad \eta = q_1 \partial_1 + \dots + q_n \partial_n, \quad q_i = q_i(x_{i+1}, \dots, x_n)$$

189 By Theorem 3.2, we may assume that $\eta = \delta_\nu \neq 0$ is a trivial derivation as in (3.7). By
 190 (1.2) and (3.8), we may further assume that $\nu_i = i + 1$ for all $i = 0, \dots, t$ and hence
 191 $q_i = 0$ for all $i > t + 1$. The preceding arguments combined with (3.8) and (3.9), can be
 192 summarized as follows.

193 **Lemma 3.5.** *Let A be a quasihomogeneous normal ICIS. Then, for all $\eta \in \Theta_{A,<0}$ and
 194 all $j = 1, \dots, t$, we have*

$$(3.13) \quad \eta g_j = 0.$$

195 If $\Theta_{A,<0} \neq 0$ then there is a derivation $0 \neq \eta \in \Theta_{A,<0}$ as in (3.12) with $q_i = 0$ for all
 196 $i > t$. It gives rise to an inequality

$$(3.14) \quad p_1 + \dots + p_t < w_1 + \dots + w_{t+1}. \quad \square$$

197 *Remark 3.6.* For degree reasons (see (1.2)), the identity (3.13) holds true for any $\eta \in$
 198 $\Theta_{A, < p_t - p_1}$ and any quasihomogeneous singularity A as in (1.1).

199 We now link the conditions $\mathfrak{A}(k)$ and $\mathfrak{B}(k)$ from page 4 to the existence of a negative
 200 derivation.

201 **Lemma 3.7.** *Assume that $\eta \in \Theta_{A, < 0}$ as in (3.12) with $q_i = 0$ for all $i \in I \supset \{0, \dots, k -$
 202 $1\}$ satisfies (3.13) for all $j = 1, \dots, t$ and that $\mathfrak{A}(k)$ holds true. Then there is a χ -
 203 homogeneous coordinate change preserving the preceding conditions which makes $q_k = 0$.*

204 *Proof.* As $\mathfrak{A}(k)$ holds by hypothesis, there is a $g := g_j$ containing a monomial x_k^m , $m > 1$.
 205 Expanding respect to x_k ,

$$g = \sum_{j=0}^m t_j x_k^{m-j}, \quad t_j = t_j(x_1, \dots, \widehat{x}_k, \dots, x_n).$$

We may normalize g such that $t_0 = \frac{1}{m}$. Note that t_j is homogeneous of degree $j \cdot w_k$ and,
 in particular, $\deg(t_1) = \deg(x_k)$. Then, ordering terms according to $i = k$ or $i > k$, (3.13)
 becomes

$$(3.15) \quad 0 = \eta(g) = \sum_{i \geq k} \sum_{j=0}^m q_i \partial_i (t_j x_k^{m-j}) \\ = \sum_{j=1}^m \left((m-j+1) q_k t_{j-1} + \sum_{i > k} q_i \partial_i (t_j) \right) x_k^{m-j}.$$

206 By (3.12), all q_i , $i \geq k$, are independent of x_k . Thus, using $t_0 = \frac{1}{m}$, the coefficient
 207 equation of x_k^{m-1} in (3.15) reads

$$q_k + \sum_{i > k} q_i \partial_i (t_1) = 0$$

208 and η can be rewritten as

$$(3.16) \quad \eta = \sum_{i > k} q_i \cdot (\partial_i - \partial_i(t_1) \partial_k).$$

209 The χ -homogeneous coordinate change

$$x'_i = \begin{cases} x_k + t_1, & \text{if } i = k, \\ x_i, & \text{otherwise,} \end{cases}$$

210 replaces $\partial_i - \partial_i(t_1) \partial_k$ in (3.16) by ∂'_i , and thus q_k in (3.12) by 0. □

211 Our main technical result is the following

212 **Proposition 3.8.** *Let A be a quasihomogeneous normal ICIS such that $\Theta_{A, < 0} \neq 0$. Then
 213 $\mathfrak{B}(k)$ holds true for some $k \leq t$ after some χ -homogeneous coordinate change. Each such
 214 k satisfies $k \geq t - d + 2$ and $g_k, \dots, g_t \notin \mathfrak{m}_P^3$.*

215 *Proof.* Let $0 \neq \eta \in \Theta_{A, < 0}$ be given by Lemma 3.5. For increasing $k \geq 1$ with $q_k \neq 0$, we
 216 repeatedly apply Lemma 3.7 with $I = \{1, \dots, k - 1, t + 2, \dots, n\}$ as long as $\mathfrak{A}(k)$ holds.
 217 The procedure stops with

$$(3.17) \quad 0 \neq \eta = q_k \partial_k + \dots + q_{t+1} \partial_{t+1}, \quad q_k \neq 0,$$

218 for some $k \leq t + 1$ by choice of η and Lemma 3.7. Since $\mathfrak{A}(k)$ fails, $\mathfrak{B}(k)$ must hold true.
 219 In case $k = t + 1$ in (3.17), (3.13) becomes $\partial_k g_j = 0$ for all $j = 1, \dots, t$. This would mean

220 that all g_j are independent of x_k in contradiction to the isolated singularity hypothesis.

221 Therefore, $k \leq t$ as claimed.

222 Combining (2.6) and (3.14), we obtain

$$(3.18) \quad p_j + \cdots + p_t + jw_n \leq w_j + \cdots + w_{t+1}$$

for all $j = 1, \dots, t$. Using (1.2), $\mathfrak{B}(k)$ and (3.18) for $j = k$, we compute

$$\begin{aligned} m_k w_k + \cdots + m_t w_t &\leq (m_k + \cdots + m_t) w_k \\ &= \deg(\partial_{\nu_k} g_k \cdots \partial_{\nu_t} g_t) \\ &= p_k + \cdots + p_t - w_{\nu_k} - \cdots - w_{\nu_t} \\ &\leq w_k + \cdots + w_{t+1} - w_{\nu_k} - \cdots - w_{\nu_t} - k w_n. \end{aligned}$$

223 and hence

$$(m_k - 1)w_k + \cdots + (m_t - 1)w_t \leq w_{t+1} - w_{\nu_k} - \cdots - w_{\nu_t} - k w_n \leq w_{t+1}.$$

By (1.2), the left-hand side must be zero and hence

$$(3.19) \quad \begin{aligned} m_k = \cdots = m_t &= 1, \\ w_{t+1} &\geq w_{\nu_k} + \cdots + w_{\nu_t} + k w_n. \end{aligned}$$

224 In particular,

$$(3.20) \quad \nu_k, \dots, \nu_t \geq t + 2$$

225 and hence $k \geq t - d + 2$. □

226

4. ICIS OF EMBEDDING DIMENSION 5

227 **Lemma 4.1.** *Let A be a quasihomogeneous normal ICIS such that $\Theta_{A, <0} \neq 0$. Then*
 228 *$\mathfrak{A}(k_1)$ and $\mathfrak{B}(k_2)$ for $\{k_1, k_2\} = \{1, 2\}$ is impossible.*

229 *Proof.* Assuming the contrary, one of the g_j has a monomial divisible by $x_{k_1}^2$ by $\mathfrak{A}(k_1)$
 230 and each of the g_j has a monomial divisible by x_{k_2} by $\mathfrak{B}(k_2)$. In particular,

$$p_1 + \cdots + p_t \geq 2w_{k_1} + (t-1)w_{k_2} \geq w_1 + \cdots + w_{t+1}$$

231 contradicting (3.14). □

232 **Proposition 4.2.** *For any quasihomogeneous ICIS A as in (1.1) with $n = 5$ and $t = 2$,*
 233 *we have $\Theta_{A, <0} = 0$.*

Proof. Assume that $\Theta_{A, <0} \neq 0$. By Proposition 3.8 and Lemma 4.1, we must have $\mathfrak{B}(1)$
 and $\mathfrak{B}(2)$. Using (1.2), (3.19), and (3.20), we may write

$$\begin{aligned} g_1 &= x_1 x_4 + c_1 x_2^j x_{k_1} + \cdots \\ g_2 &= x_1 x_5 + c_2 x_2 x_{k_2} + \cdots \end{aligned}$$

234 with $\{k_1, k_2\} = \{4, 5\}$ and $c_1, c_2 \in K^*$. As in the proof of Lemma 4.1, the inequality (3.14)
 235 can only hold true if $j = 1$. In this case,

$$A/(J_A + \langle x_3, \dots, x_n \rangle) = K \langle \langle x_1, x_2 \rangle \rangle / \left\langle \left| \frac{\partial g}{\partial(x_4, x_5)} \right| \right\rangle.$$

236 for degree reasons (see (1.2)), and hence J_A is not \mathfrak{m}_A -primary. This contradicts to the
 237 isolated singularity hypothesis. □

239 *Proof of Example 1.2.* The sequence g is clearly regular and defines a complete intersec-
 240 tion as in (1.1). Note that η in (1.4) agrees with $\eta = \delta_{1,2,3}$ in (3.12). Since $\deg(g_1) =$
 241 $10 = \deg(g_2)$, (3.9) shows that η has negative degree $\deg \eta = -1$.

242 It remains to check that A has an isolated singularity, that is, the Jacobian ideal J_A
 243 from (2.4) is \mathfrak{m}_A -primary. To this end, we may assume that $K = \bar{K}$ which enables us to
 244 argue geometrically on the variety

$$\bar{X} := \text{Spec } \bar{A} \subset \mathbb{A}_K^n$$

245 with \bar{A} as in (2.2) using the Nullstellensatz.

246 The ideal J_A is the image in A of the Jacobian ideal $\bar{J}_g \subseteq \bar{P}$ of g generated by the
 247 2×2 -minors

$$M_{i,j} := \left| \frac{\partial g}{\partial(x_i x_j)} \right|$$

248 of the Jacobian matrix of g which reads

$$\frac{\partial g}{\partial x} = \begin{pmatrix} x_4 & x_5 & 2x_3 & x_1 - 5x_4^4 & x_2 & 0 & 5x_7^4 & \cdots & 5x_n^4 \\ x_5 & x_6 & 2x_3 & 0 & x_1 & x_2 + 5x_6^4 & 5c_7x_7^4 & \cdots & 5c_nx_n^4 \end{pmatrix}.$$

249 With this notation we have to show that

$$\text{Sing } \bar{X} = V(g, \bar{J}_g) = \{0\}.$$

250 Due to those 2×2 -minors of $\frac{\partial g}{\partial x}$ which involve only the columns 3, 7, 8, 9, \dots , n , only one
 251 of components $x_3, x_7, x_8, x_9, \dots, x_n$ of any $x \in \text{Sing } \bar{X}$ can be non-zero. We may therefore
 252 reduce to the case $n \leq 7$.

253 Because of the 3rd column of $\frac{\partial g}{\partial x}$, we have $\bar{J}_g \cap K[x_1, \dots, x_6] \supseteq x_3I$ where

$$I := \langle x_4 - x_5, x_5 - x_6, x_1 - x_2, x_1 - 5x_4^4, x_2 + 5x_6^4 \rangle.$$

254 Note that $V(I)$ is the x_3 -axis which is not contained in $V(g)$. It follows that $\text{Sing } \bar{X} \cap V(x_7)$
 255 is contained in the hyperplane $V(x_3)$. Similarly because of the 7th column of $\frac{\partial g}{\partial x}$ and
 256 setting $c := c_7$, we have $\bar{J}_g \cap K[x_1, \dots, \hat{x}_3, \dots, x_7] \supseteq x_7I'$ where

$$I' := \langle cx_4 - x_5, cx_5 - x_6, cx_2 - x_1, x_1 - 5x_4^4, x_2 + 5x_6^4 \rangle.$$

257 Using $c^9 + 1 \neq 0$, we find that $V(I')$ is the x_7 -axis and conclude $\text{Sing } \bar{X} \cap V(x_3) \subset V(x_7)$
 258 as before. Summarizing the two cases, $\text{Sing } \bar{X}$ is in fact contained in $V(x_3, x_7)$.

Fix a point $(x_1, x_2, 0, x_4, x_5, x_6, 0) \in \text{Sing } \bar{X}$. Successively using the the equations

$$\begin{aligned} M_{1,2} &= x_4x_6 - x_5^2 = 0, \\ M_{2,5} &= x_1x_5 - x_2x_6 = 0, \\ g_2 &= x_1x_5 + x_2x_6 + x_6^5 = 0, \\ M_{4,5} &= x_1(x_1 - 5x_4^4) = 0, \\ M_{5,6} &= x_2(x_2 + 5x_6^4) = 0, \end{aligned}$$

259 we derive

$$x_4 = 0 \Rightarrow x_5 = 0 \Rightarrow x_2x_6 = 0 \Rightarrow x_6 = 0 \Rightarrow x_1 = x_2 = 0.$$

260 Similarly $x_6 = 0$ leaves no possibility except $x = 0$ and $x_5 = 0$ reduces to one of these
 261 two cases by $M_{1,2} = 0$.

262 Assume now that x_4, x_5, x_6 are all non zero. Then the minors $M_{1,5}, M_{2,4}, M_{2,5}, M_{2,6}$
 263 give equations

$$x_1x_4 = x_2x_5, \quad x_1 = 5x_4^4, \quad x_1x_5 = x_2x_6, \quad x_2 = -5x_6^4.$$

264 Substituting into g , we obtain

$$g_1 = 2x_1x_4 - x_4^5 = 9x_4^5, \quad g_2 = 2x_2x_6 + x_6^5 = -9x_6^5$$

265 and hence $x_4 = x_6 = 0$ contradicting our assumption. \square

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