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Maximal multihomogeneity of algebraic hypersurface singularities

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Abstract From the degree zero part of the logarithmic vector fields along an algebraic hypersurface singularity we identify the maximal multihomogeneity of a defining equation in form of a maximal algebraic torus in the embedded automorphism group. We show that all such maximal tori are conjugate and in one–to–one correspondence to maximal tori in the linear jet of the embedded automorphism group.

These results are motivated by Kyoji Saito's characterization of quasihomogeneity for isolated hypersurface singularities [Sai71] and extend previous work with Granger [GS06a, Thm. 5.4] and of Hauser and Müller [HM89, Thm. 4].

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1 Introduction and notation

A convergent power series $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ with $f(0) = 0$ defines a complex hypersurface singularity $X = \{x \in \mathbb{C}^n \mid f(x) = 0\}$ in the space germ $(\mathbb{C}^n, 0)$ of $\mathbb{C}^n$ at the origin. The singularity is isolated if the gradient ideal $J(f) = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle$ of f defines the origin, that is, $\{0\} = \left\{x \in \mathbb{C}^n \mid \frac{\partial f}{\partial x_1}(x) = \cdots = \frac{\partial f}{\partial x_n}(x) = 0\right\}$. In this case, there is a $\kappa \geq 1$ such that f^κ is contained in $J(f)$, that is, $f^\kappa = g_1 \frac{\partial f}{\partial x_1} + \cdots + g_n \frac{\partial f}{\partial x_n}$ for some $g_1, \ldots, g_n \in \mathbb{C}\{x_1, \ldots, x_n\}$. The power series f is weakly quasihomogeneous of degree $w \in \mathbb{Z}$ with respect to a weight vector $0 \neq (w_1, \ldots, w_n) \in \mathbb{Z}^n$ if $f(x_1, \ldots, x_n) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha x^\alpha$ where $f_\alpha \neq 0$ only if $w_1\alpha_1 + \cdots + w_n\alpha_n = w$ [Sai71, §1]. While this condition depends on coordinates $x_1, \ldots, x_n$, it implies for $w \neq 0$ that $\kappa = 1$ by choosing $g_i = \frac{w_i}{w} x_i$ for $i = 1, \ldots, n$. If f has order at least 3 then the isolatedness of the singularity forces $w, w_1, \ldots, w_n \in \mathbb{Z}_{>0}$ [Sai71, Lem. 1.10]. Strictly positive weights reduce a power series to a polynomial which is then called a quasihomogeneous polynomial. In his celebrated article [Sai71], Saito proves the converse of the above

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implication: If $\kappa = 1$ then $g_1(0) = \cdots = g_n(0) = 0$ [Sai71, Lem. 4.2] and there is a coordinate system $x_1, \dots, x_n$ in which $f(x_1, \dots, x_n)$ is a quasihomogeneous polynomial [Sai71, Satz]. Our results can be considered as a generalization of Saito's theorem for algebraic hypersurface singularities. Instead of a single homogeneity, we shall identify multihomogeneities of the singularity.

The study of (multi–)homogeneities of singularities is of interest for its topological implications: For example, Saito shows that quasihomogeneity of X is equivalent to holomorphic contractibility of X and to the exactness of the Poincaré complex of X [Sai71, Satz]. Another example is the problem of characterizing hypersurfaces for which the so–called logarithmic comparison theorem holds [Tor07]. In analogy to Grothendieck's famous comparison theorem [Gro66], this "theorem", with yet undetermined hypothesis, states that the complex cohomology of the complement of a hypersurface in a complex manifold is the hypercohomology of the logarithmic de Rham complex. This statement holds, for example, for normal crossing divisors and, more generally, for strongly quasihomogeneous free divisors [CJNMM96]. A central conjecture by Calderón–Moreno et al. [GS06a, Cnj. 1.1] states that, for free divisors, the logarithmic comparison theorem requires a homogeneity condition called strong Euler homogeneity. In the above notation, a hypersurface X is strongly Euler homogeneous if it admits at each point $p = (p_1, \dots, p_n) \in X$ a defining equation $f \in \mathbb{C}\{x_1 - p_1, \dots, x_n - p_n\}$ with $\kappa = 1$ and $g_1(p) = \cdots = g_n(p) = 0$. The algebraicity hypothesis in our results is fulfilled in particular for the class of linear free divisors which form an interesting class of examples to test the mentioned conjecture [GMNS06]. Even for isolated singularities the characterization of the logarithmic comparison theorem is incomplete [GS06b].

In order to further motivate and finally state our results we introduce the terminology that we shall use. Let $\mathcal{P}$ be either $\mathcal{O} = \mathbb{C}\{x\}$, the ring of convergent power series in $x = x_1, \dots, x_n$, or $\hat{\mathcal{O}} = \mathbb{C}[\![x]\!]$, the ring of formal power series in x and denote by $\mathfrak{m} = \langle x \rangle$ the maximal ideal in $\mathcal{O}$ or $\hat{\mathcal{O}}$. Let $\mathrm{Aut}(\mathcal{P})$ be the automorphism group of $\mathcal{P}$, $\mathrm{Der}(\mathcal{P})$ the $\mathcal{P}$-module of $\mathbb{C}$-linear derivations on $\mathcal{P}$, and denote $\Delta(\mathcal{P}) = \mathfrak{m} \cdot \mathrm{Der}(\mathcal{P}) = \{\delta \in \mathrm{Der}(\mathcal{P}) \mid \delta(\mathfrak{m}) \subseteq \mathfrak{m}\}$. In the convergent case, the latter consists of those holomorphic vector fields on $(\mathbb{C}, 0)$ that vanish at the origin. The commutator of two vector fields $\delta_1, \delta_2 \in \mathrm{Der}(\mathcal{P})$ is denoted by $[\delta_1, \delta_2] \in \mathrm{Der}(\mathcal{P})$.

Let $0 \neq f \in \mathfrak{m}$ and $\mathrm{Aut}_f = \{\varphi \in \mathrm{Aut}(\mathcal{P}) \mid \varphi(f) \in \langle f \rangle\}$ the group of automorphisms preserving the ideal $\langle f \rangle$. In the convergent case, this is the group of automorphisms of $(\mathbb{C}^n, 0)$ preserving the hypersurface X. Our object of interest is the $\mathcal{P}$-module of logarithmic vector fields $\mathrm{Der}_f = \{\delta \in \mathrm{Der}(\mathcal{P}) \mid \delta(f) \in \langle f \rangle\}$ introduced in [Sai80]. In the convergent case, this is the module of vector fields tangent to the smooth part of X. The module Der_f is unchanged if we assume f to be reduced. We shall further assume that $\mathrm{Der}_f \subseteq \Delta(\mathcal{P})$. By Rossi's theorem [Ros63, Cor. 3.4], this means in the convergent case that the variety X defined by f is not a product with a smooth factor. As remarked in [HM93, 2. Rmk. (c)], Der_f is the Lie algebra of the infinite Lie group Aut_f in the convergent case. If $\chi(f) \in \mathbb{C}^* \cdot f$ then $\chi \in \Delta(\mathcal{P})$ is called an Euler vector field. Strong Euler homogeneity of a hypersurface X can be reformulated as the existence of an Euler vector field at each $p \in X$.

For a fixed coordinate system, any derivation $\delta \in \mathrm{Der}(\mathcal{P})$ can be decomposed into homogeneous components, $\delta = \sum_{i=-1}^{\infty} \delta_i$. Moreover, $\delta_0 = \sum_{i,j} a_{i,j} x_i \partial_{x_j}$ for some matrix $A = (a_{i,j})$ and we call δ_0 diagonal if A is diagonal. A derivation $\delta \in \Delta(\mathcal{P})$ is called nilpotent if A is nilpotent and semisimple if $\mathfrak{m}$ has a basis of eigenvectors of δ. For a fixed coordinate system, any $\delta \in \Delta(\mathcal{P})$ is a sum $\delta = \delta_S + \delta_N$ where $\delta_S = \delta_{S,0}$ and $\delta_{N,0}$ are defined by the semisimple and nilpotent parts of the matrix A corresponding to δ_0.

In particular, $[\delta_S, \delta_{N,0}] = 0$. One can construct a formal coordinate change such that even $[\delta_S, \delta_N] = 0$ and then $\delta = \delta_S + \delta_N$ is called the Poincaré–Dulac decomposition of δ [AA88, Ch. 3. §3.2].

We return to Saito's theorem: Reformulated in the introduced terminology, $\chi = \sum_{i=1}^n g_i \partial_{x_i}$ is an Euler vector field and $\sigma = \sum_{i=1}^n w_i x_i \partial_{x_i}$ is a diagonal Euler vector field with $\sigma(f) = w \cdot f$. In his proof, Saito applies Artin approximation theorem [Art68, Thm. 1.2] to the formal Poincaré–Dulac coordinate change [Sai71, Satz 3] in order to obtain a convergent coordinate change. The strict positivity of the weights $w_1, \ldots, w_n$ derived from the isolatedness assumption is crucial in this procedure. A more detailed examination of his proof shows that σ comes from χ_S via the linear part of this coordinate change if the order of f is at least 3. By integration, σ corresponds to the one–parameter subgroup $\{\varphi_\lambda \mid \lambda \in \mathbb{C}^*\} \subseteq \mathrm{Aut}(\mathcal{O})$ defined by $\varphi_\lambda(x_i) = \lambda^{w_i} x_i$. Assuming $w_1, \ldots, w_n$ to be coprime, $\lambda \mapsto \varphi_\lambda$ defines an inclusion $\mathbb{C}^* \subseteq \mathrm{Aut}(\mathcal{O})$. As $\varphi_\lambda(f) = \lambda^w \cdot f$, the latter inclusion factors through $\mathrm{Aut}_f \subseteq \mathrm{Aut}(\mathcal{O})$. The constructed coordinate system makes this 1-torus $\mathbb{C}^*$ explicitly visible as a diagonal subgroup of the group of linear transformations $\mathrm{GL}_n(\mathbb{C}) \subseteq \mathrm{Aut}_f$ defined by the coordinate system.

Various attempts have been made to generalize Saito's result. While Scheja and Wiebe [SW77] drop the hypersurface assumption and treat the case of isolated complete intersection singularities, Hauser and Müller [HM89, Thm. 4] drop the isolatedness assumption and consider algebraic singularities. In the latter approach the approximation property of excellent Henselian local rings due to Popescu and Rotthaus replaces the classical Artin approximation. The above discussion suggests another direction of generalization: d-dimensional vector spaces of commuting semisimple parts δ_S of logarithmic vector fields $\delta \in \mathrm{Der}_f$ should define complex d-tori $(\mathbb{C}^*)^d$ in Aut_f which we interpret as multihomogeneities of the singularity X. In the formal case, this follows from Part a of Theorem 1 which is a reformulation of the formal structure theorem for Der_f in [GS06a, Thm. 5.4]. Part b of Theorem 1 is a convergent version of this structure theorem for algebraic hypersurface singularities. While the formulation in [GS06a, Thm. 5.4] bypasses the uniqueness question for maximal multihomogeneities, we establish this uniqueness up to conjugacy in Theorem 2 using general results of Müller [Mül86] on automorphism groups of singularities. Moreover, we conclude in Corollary 1 that the maximal multihomogeneities of X correspond to maximal tori in the linear jet of Aut_f.

2 Results and proofs

Part a of the following theorem is a reformulation of the formal structure theorem for Der_f in [GS06a, Thm. 5.4] and the statement a.5 is implicit in its proof. We shall deduce Part b from Part a following the outline of [HM89, Thm. 4].

Theorem 1 (existence of maximal multihomogeneity) *Let*

a. $f \in \hat{\mathcal{O}}$ or

b. $f \in \mathcal{O}$ algebraic over $\mathbb{C}[x]$

and let $\delta_1, \ldots, \delta_t \in \mathrm{Der}_f$ with diagonal degree 0 part. Then there is an algebraic torus $\mathrm{T}^s \subseteq \mathrm{Aut}_f$ in the sense of [Mül86, 1. Def. ii)] with Lie algebra $\mathfrak{t}^s$ and suitable coordinates such that $\mathrm{T}^s \subseteq \mathrm{GL}_n(\mathbb{C})$. In these coordinates there is a basis $\sigma_1, \ldots, \sigma_s$ of $\mathfrak{t}^s$ which extends to a minimal system of generators $\sigma_1, \ldots, \sigma_s, \nu_1, \ldots, \nu_r$ of Der_f and a choice of f with irreducible factors $f_1, \ldots, f_m$ such that

1. *σ_i is diagonal with eigenvalues in $\mathbb{Z}$,*
2. *ν_i is nilpotent,*
3. *$[\sigma_i, \nu_j] \in \mathbb{Z} \cdot \nu_j$,*
4. *$\sigma_i(f_j) \in \mathbb{Z} \cdot f_j$,*
5. *$(\delta_i)_0 \in \langle \sigma_1, \ldots, \sigma_s \rangle_{\mathbb{C}}$, and*
6. *if $\delta \in \mathrm{Der}_f$ with $[\sigma_i, \delta_0] = 0$ for all i then $\delta_S \in \langle \sigma_1, \ldots, \sigma_s \rangle_{\mathbb{C}}$.*

Proof The statement a.5 holds by the construction in the proof of [GS06a, Thm. 5.4] using that the coordinate change in [GS06a, Thm. 5.3] is tangent to the identity.

By a.1 and a.4, there is a formal coordinate change $\bar{y}(x)$ such that $\bar{g}(x) = f(\bar{y}(x))$ fulfills $\sigma_i(\bar{g}) = \lambda_i \cdot \bar{g}$ where $\lambda_i \in \mathbb{Z}$ and $\sigma_i = \sum_{j=1}^n \sigma_{i,j} x_j \partial_{x_j}$ where $\sigma_{i,j} \in \mathbb{Z}$. We identify σ_i with its coefficient vector $(\sigma_{i,1}, \ldots, \sigma_{i,n}) \in \mathbb{Z}^n$. By [Bor91, II.7.3 (2)], the saturation

$$L = ((\mathbb{Z}^n)^\vee / (\mathbb{Z}^n / \sum_{i=1}^s \mathbb{Z}\sigma_i)^\vee)^\vee \cong \mathbb{Z}^s \subseteq \mathbb{Z}^n$$

of the lattice generated by $\sigma_1, \ldots, \sigma_s$ defines an algebraic torus

$$\mathrm{G}_m^s \cong \mathrm{T}^s := \mathrm{Spec}(\mathbb{C}[L]) \subseteq \mathrm{Spec}(\mathbb{C}[\mathbb{Z}^n]) \subseteq \mathrm{GL}_n(\mathbb{C}) \subseteq \mathrm{Aut}(\mathcal{P})$$

with Lie algebra $\mathfrak{t}^s = \langle \sigma_1, \ldots, \sigma_s \rangle$. Since the x-monomials are common eigenvectors for $x_1\partial_1, \ldots, x_n\partial_n$ with integer eigenvalues, we may assume $L = \sum_{i=1}^s \mathbb{Z}\sigma_i$ preserving the condition $\lambda_i \in \mathbb{Z}$. Then the σ-multihomogeneity of $\bar{g}$ of multidegree λ stated above translates to $\bar{g}$ being equivariant for $\mathrm{T}^s \subseteq \mathrm{GL}_n(\mathbb{C})$ and the character $\mathrm{T}^s \to \mathrm{Spec}(\mathbb{C}[\mathbb{Z}]) = \mathrm{GL}_1(\mathbb{C})$ defined by $L \ni \sigma_i \mapsto \lambda_i \in \mathbb{Z}$. Now $\langle f \rangle_{\hat{\mathcal{O}}}$ is equivalent to the T^s-stable ideal $\langle \bar{g} \rangle_{\hat{\mathcal{O}}}$ and [HM89, Thm. 2'] implies that also $\langle f \rangle_{\mathcal{O}}$ is equivalent to a T^s-stable ideal. This means that there is an analytic coordinate change $y(x)$ and a unit $u \in \mathcal{O}^*$ such that $g(x) = u(x) \cdot f(y(x))$ generates a T^s-stable ideal in $\mathcal{O}$. By abuse of notation we denote this g by f again. Then [Mül86, Hilfssatz 2] shows that $\langle f \rangle_{\mathcal{O}}$ has a T^s-equivariant generator which we may assume to be f. This proves b.1 and $\sigma_i(f) \in \mathbb{Z} \cdot f$ which is weaker than b.4.

To avoid the non-trivial [Mül86, Hilfssatz 2] and conclude b.4, one can argue as follows: By [Hum75, Thm. 13.2], the T^s- and $\mathfrak{t}^s$-stable subspaces in $\mathcal{O}/\mathfrak{m}^k$ coincide. Since ideals in $\mathcal{O}$ are closed in the $\mathfrak{m}$-adic topology, this shows that $\langle f \rangle_{\mathcal{O}}$ is σ-stable. Then a multigraded version[1] of [SW73, 2.4] shows that $\langle f \rangle_{\mathcal{O}}$ has a T^s-equivariant generator. With $\langle f \rangle_{\mathcal{O}}$ also its minimal associated primes are σ-stable by [SW73, 2.5]. Again by [SW73, 2.4], these primes have T^s-equivariant generators $f_1, \ldots, f_m$ and b.4 follows.

With f also its partial derivatives $\partial_{x_i}(f)$ are T^s-equivariant. Thus Der_f is T^s-stable since it can be considered as the projection of the syzygy module of $\partial_{x_1}(f), \ldots, \partial_{x_n}(f), f$ to the first n components. Its T^s-sub-module $\mathfrak{t}^s = \langle \sigma_1, \ldots, \sigma_s \rangle$ is the adjoint representation of T^s and hence trivial. By a module version[2] of [Mül86, Hilfssatz 2], Der_f has a minimal system of generators $\sigma_1, \ldots, \sigma_s, \nu_1, \ldots, \nu_r$ which spans over $\mathbb{C}$ a rational T^s-module. By the proof of [Mil06, Thm. 9.13], the T^s-module $\langle \sigma_1, \ldots, \sigma_s, \nu_1, \ldots, \nu_r \rangle_{\mathbb{C}}$ can be diagonalized without changing the σ_i and b.3 follows. Like above, one can use a multigraded module version of [SW73, 2.4] instead.

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[1] The arguments in [SW73, 2.2-4] give a more general correspondence of $(k^s, +)$-graduations and sets of s simultaneously diagonalizable k-derivations on analytic k-algebras.

[2] The statement in [Mül86, Hilfssatz 2] holds more generally for any analytic sub-module of a free analytic module in the sense of [GR71].

The maximality property b.6 follows from its formal version a.6. Combined with b.3 it guarantees the existence of ν_i satisfying b.2: If the σ-multidegree of a ν_i is non-zero then it is nilpotent by [GS06a, Lem. 2.6], otherwise one can subtract its semisimple part which is a linear combination of the σ_i by b.6.

Remark 1

1. By [KPR75, 2.11], the implicit function theorem holds in $\mathbb{C}\langle x\rangle$, the ring of algebraic power series. Thus the proof of [HM89, Thm. 2'] works inside of $\mathbb{C}\langle x\rangle$ as well. However it is not clear if this ring is graded in the sense of [SW73, §1-2]. A positive answer would imply that Theorem 1.b even holds inside of $\mathbb{C}\langle x\rangle$.

2. According to Michel Granger, the uniqueness of s in Theorem 1 can be deduced from the conjugacy of all Cartan subalgebras [Ser01, III.4. Thm. 2] by showing that $\mathfrak{t}^s$ and $\mathfrak{n}_0 = \langle(\nu_j)_0 \mid \sigma_i(\nu_j) = 0\rangle_{\mathbb{C}}$ span a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g} = \mathrm{Der}_f / (\mathrm{Der}_f \cap \mathfrak{m}^2 \cdot \mathrm{Der}(\mathcal{P}))$ where $\mathfrak{n}_0$ is the intersection of $\mathfrak{h}$ with the set of nilpotent elements in $\mathfrak{g}$.

We shall use results of Müller [Mül86] to prove a stronger statement than uniqueness of s. Consider the group morphisms $\pi_k : \mathrm{Aut}(\mathcal{P}) \to \mathrm{Aut}_k(\mathcal{P}) = \mathrm{Aut}(\mathcal{P}/\mathfrak{m}^{k+2})$ and the Lie algebra morphisms $\pi_k : \Delta(\mathcal{P}) \to \Delta_k(\mathcal{P}) = \Delta(\mathcal{P})/(\mathfrak{m}^{k+1}\cdot\Delta(\mathcal{P}))$. Note that $\mathrm{Aut}_k(\mathcal{P})$ is an algebraic group with Lie algebra $\Delta(\mathcal{P})$. Like in [Mül86, §2], one can use Artin approximation theorem [Art68, Thm. 1.2] to prove the first part of the following

Lemma 1 $\pi_k(\mathrm{Aut}_f)$ *is an algebraic group with Lie algebra* $\pi_k(\mathrm{Der}_f)$*.*

Proof By exactness of completion $\mathrm{Der}_{\hat f} = \widehat{\mathrm{Der}_f}$ where $\hat f$ denotes f considered in $\hat{\mathcal{O}}$. Thus $\pi_k(\mathrm{Der}_f) = \pi_k(\mathrm{Der}_{\hat f})$ and we may assume that $\mathcal{P} = \hat{\mathcal{O}}$. Consider the Lie algebra morphisms $\pi_k^m : \Delta_m(\mathcal{P}) \to \Delta_k(\mathcal{P})$ and denote by f_k the image of f in $\mathcal{O}/\mathfrak{m}^{k+1}$. For fixed k, the $\pi_k^m(\mathrm{Der}_{f_m})$ form a decreasing sequence of sub vector spaces in $\Delta_k(\mathcal{P})$ which implies $\pi_k^m(\mathrm{Der}_{f_m}) = D_k$ for large m. Then $\pi_m^{m+1} : D_{m+1} \to D_m$ is surjective and hence $D_k = \pi_k(\mathrm{Der}_f)$. In the proof of the first statement, we find $\pi_k^m(\mathrm{Aut}_{f_m}) = \pi_k(\mathrm{Aut}_f)$ for large m and Der_{f_m} is the Lie algebra of Aut_{f_m}.

Theorem 2 (uniqueness of maximal multihomogeneity) *The algebraic torus* T^s *in Theorem 1 is maximal in* Aut_f *and also* $\pi_0(\mathrm{T}^s) \subseteq \pi_0(\mathrm{Aut}_f)$ *is a maximal algebraic torus. All maximal algebraic tori in* Aut_f *and* $\pi_0(\mathrm{Aut}_f)$ *are conjugate.*

Proof First let $\mathrm{T}^t \subseteq \pi_0(\mathrm{Aut}_f)$ be an algebraic torus with Lie algebra $\mathfrak{t}^t$. If $\pi_0(\mathrm{T}^s) \subseteq \mathrm{T}^t$ then the Lie algebra of T^t consists of σ-homogeneous semisimple elements of multidegree 0 and hence $\pi_0(\mathfrak{t}^s) = \mathfrak{t}^t$ by Theorem 1.6. By Lemma 1 and [Mil06, Prp. 13.11], this implies $\pi_0(\mathrm{T}^s) = \mathrm{T}^t$ and hence $\pi_0(\mathrm{T}^s)$ is maximal.

Now let $\mathrm{T}^t \subseteq \mathrm{Aut}_f$ be an algebraic torus in the sense of [Mül86, 1. Def. ii)] with Lie algebra $\mathfrak{t}^t$. By definition, $\pi_k(\mathrm{T}^t) \subseteq \pi_k(\mathrm{Aut}_f) \subseteq \mathrm{Aut}_k(\mathcal{P})$ defines a rational representation of T^t which is diagonalizable by [Mil06, Thm. 9.13]. Then Cartan's uniqueness theorem holds for T^t by [Kau67, Satz] and π_0 restricted to T^t is injective. This shows that maximality of $\pi_0(\mathrm{T}^s)$ implies maximality of T^s.

By [Hum75, VIII.21.3. Cor. A], all maximal tori of $\pi_0(\mathrm{Aut}_f)$ are conjugate. An embedded version[3] of [Mül86, Satz 2] shows that a conjugacy in $\pi_0(\mathrm{Aut}_f)$ of algebraic tori in Aut_f lifts to a conjugacy in Aut_f.

[3] In the first paragraph of [Mül86, §5], the proof of [Mül86, Satz 2] is reduced to the case of embedded automorphism groups using [Mül86, Satz 6].

Corollary 1 (lifting of maximal tori) $\pi_0 : \mathrm{Aut}_f \to \pi_0(\mathrm{Aut}_f)$ *defines a bijection of algebraic tori.*

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